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GRAND UNIFICATION IN NON–ASSOCIATIVE GEOMETRY

RAIMAR WULKENHAAR

Institut für Theoretische Physik, Universität Leipzig
Augustusplatz 10/11, D–04109 Leipzig, Germany
E-mail: raimar.wulkenhaar@itp.uni-leipzig.de

Abstract. We formulate the flipped $\mathrm{SU}(5) \times \mathrm{U}(1)$–GUT within the framework of non–associative geometry. It suffices to take the matrix Lie algebra su(5) as the input; the u(1)–part with its representation on the fermions is an algebraic consequence. The occurring Higgs multiplets ($\underline{24}, \underline{5}, \underline{45}, \underline{50}$–representations of su(5)) are uniquely determined by the fermionic mass matrix and the spontaneous symmetry breaking pattern to $\mathrm{SU}(3)_C \times \mathrm{U}(1)_{EM}$. We find the most general gauge invariant Higgs potential that is compatible with the given Higgs vacuum. Our formalism yields tree–level predictions for the masses of all gauge and Higgs bosons. It turns out that the low–energy sector is identical with the standard model. In particular, there exists precisely one light Higgs field, whose upper bound for the mass is $1.45\, m_t$. All remaining 207 Higgs fields are extremely heavy.

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1. Introduction

One of the most important applications of non–commutative geometry (NCG) to physics is a unified description of the standard model. The most elegant version rests upon a K–cycle [4, 6] with real structure [5], see [8, 11] for details of the construction. The standard model is the only realistic physical model that one can formulate within the most elegant NCG–prescription [10]. On the other hand, there exist good reasons [9] why one could be interested in Grand Unified Theories (GUT's): GUT's explain the quantization of electric charge, yield a fairly well prediction for the Weinberg angle, explain the convergence of running coupling constants at high energies, include massive neutrinos to solve the solar neutrino problem, produce the observed baryon asymmetry of the universe, etc. However, the results of [10] imply that one needs additional structures or different methods for a NCG–formulation of these models.

The perhaps most successful NCG–approach towards grand unification was proposed by Chamseddine, Felder and Fröhlich. In the SU(5)–model [1, 2], the authors start to construct an auxiliary K–cycle. Within this framework they construct the bosonic sector. Then they interpret some of these bosonic quantities

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as Lie algebra valued and consider Lie algebra representations on the physical Hilbert space to obtain the fermionic sector. An aesthetic shortcoming of that approach is the auxiliary character of the K–cycle, which of course is inevitable in view of [10]. The SO(10)–model [3] by Chamseddine and Fröhlich fits well[1] into the NCG–scheme. The reason why this model was excluded in [10] is that only models possessing complex fundamental irreducible representations were admitted in that article.

The author of this paper has proposed in [13] a modification of non–commutative geometry. In that approach one uses skew–adjoint Lie algebras instead of unital associative $*$–algebras. Lie algebras are non–associative algebras – this is the motivation for the working title "non–associative geometry". The advantage of non–associative geometry is that a larger class of physical models can be constructed from the same amount of structures as in the most elegant NCG–formulation. That class includes the standard model [14] and the flipped $\mathrm{SU}(5) \times \mathrm{U}(1)$–GUT as well, as we show in this paper. The SU(5)–model can be obtained as a special case. For the classical treatment of the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$–model see [7].

We give in Section 2 a recipe how to construct classical gauge field theories within non–associative geometry. The arguments why this recipe works can be found in [13]. In Section 3 we construct the matrix part of the $\mathrm{SU}(5) \times \mathrm{U}(1)$–model: In Section 3.1 we consider relevant su(5)–representations. The remaining ingredients of non–associative geometry are defined in Section 3.2. Then it is not difficult to derive in Section 3.3 the matrix part of the connection form. Finally, we perform in Section 3.6 the factorization of the curvature with respect to a canonically given ideal, which we construct before in Section 3.5.

In Section 4 we include the space–time part and derive the action for our model: Out of the curvature obtained in Section 4.1 we build in Section 4.2 the bosonic action. To compare it with usual formulae of gauge field theory we write down this action in terms of local coordinates, see Section 4.3. The fermionic action is derived in Section 4.4. Comparing it with phenomenology we can identify certain parameters of the generalized Dirac operator with fermion masses and Kobayashi–Maskawa mixing angles.

This information plays an essential rôle in deriving the masses of the Higgs bosons in Section 5. Finally, we comment on a formal derivation of the SU(5)–GUT in Section 6.

2. The Recipe of Non–associative Geometry

The basic object in non–associative geometry is an L–cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$, which consists of a $*$–representation π of a skew–adjoint Lie algebra $\mathfrak{g}$ as bounded operators on a Hilbert space h, together with a selfadjoint operator D on h with compact resolvent and a selfadjoint operator Γ on h, $\Gamma^2 = \mathrm{id}_h$, which commutes with $\pi(\mathfrak{g})$ and anticommutes with D. The operator D may be unbounded, but

[1]Nevertheless, the use of Lie algebras instead of algebras could probably justify certain assumptions made in [3].

such that $[D,\pi(\mathfrak{g})]$ is bounded. L–cycles are naturally related to physical models on a space–time manifold X if the following input data are given:

1) A unitary matrix Lie group G and its associated gauge group $\mathcal{G} = C^\infty(X)\otimes G$. Here, $C^\infty(X)$ denotes the algebra of real–valued smooth functions on X.
2) Chiral fermions $\boldsymbol{\psi}$ transforming under a representation $\tilde\pi_0$ of G. The induced representation of the gauge group $\mathcal{G}$ is $\tilde\pi = \mathrm{id}\otimes\tilde\pi_0$.
3) The fermionic mass matrix $\widetilde{\mathcal{M}}$, i.e. fermion masses plus generalized Kobayashi–Maskawa matrices.
4) Possibly the spontaneous symmetry breaking pattern of G.

For technical reasons we pass to a compact Euclidian spin manifold X. We take $\mathfrak{g} = C^\infty(X)\otimes\mathfrak{a}$ as the Lie algebra of $\mathcal{G}$. Here, $\mathfrak{a} = \mathfrak{a}'\oplus\mathfrak{a}''$ is a skew–adjoint matrix Lie algebra, where $\mathfrak{a}'$ is semisimple and $\mathfrak{a}''$ Abelian. We shall only consider the case that the Abelian part is not present, i.e. $\mathfrak{a} = \mathfrak{a}'$. We choose $h = L^2(X,S)\otimes\mathbb{C}^F$ as the space where the Euclidian fermions $\boldsymbol{\psi}$ live. Here, $L^2(X,S)$ is the Hilbert space of square integrable bispinors. We take $\pi = 1\otimes\hat\pi$ as the differential $\tilde\pi_*$, where $\hat\pi$ is a representation of $\mathfrak{a}$ in $\mathrm{M}_F\mathbb{C}$. We define $D = \mathsf{D}\otimes\mathbb{1}_F + \gamma^5\otimes\mathcal{M}$, where D is the Dirac operator associated to the spin connection and $\mathcal{M}\in\mathrm{M}_F\mathbb{C}$. Here, $\gamma^5\otimes\mathcal{M}$ has to coincides with $\widetilde{\mathcal{M}}$ on chiral fermions. The chirality properties of the fermions are encoded in $\Gamma = \gamma^5\otimes\hat\Gamma$, where $\{\hat\Gamma,\mathcal{M}\}=0$ and $[\hat\Gamma,\hat\pi(\mathfrak{a})]=0$.

The recipe towards the (classical) gauge field theory associated to the L–cycle is the following: Let $\Omega^1\mathfrak{a}$ be the space of formal commutators

$$\omega^1 = \textstyle\sum_{\alpha,z\geq 0}[a^z_\alpha,[\dots[a^1_\alpha,da^0_\alpha]\dots]]\;, \quad a^i_\alpha\in\mathfrak{a}\;. \tag{2.1}$$

Apply linear mappings $\hat\pi:\Omega^1\mathfrak{a}\to\mathrm{M}_F\mathbb{C}$ and $\hat\sigma:\Omega^1\mathfrak{a}\to\mathrm{M}_F\mathbb{C}$ defined by

$$\hat\pi(\omega^1) := \textstyle\sum_{\alpha,z\geq 0}[\hat\pi(a^z_\alpha),[\dots[\hat\pi(a^1_\alpha),[-\mathrm{i}\mathcal{M},\hat\pi(a^0_\alpha)]]\dots]]\;, \tag{2.2a}$$
$$\hat\sigma(\omega^1) := \textstyle\sum_{\alpha,z\geq 0}[\hat\pi(a^z_\alpha),[\dots[\hat\pi(a^1_\alpha),[\mathcal{M}^2,\hat\pi(a^0_\alpha)]]\dots]]\;. \tag{2.2b}$$

Define $\Omega^n\mathfrak{a}\ni\omega^n = \sum_\alpha[\omega^1_{n,\alpha},[\omega^1_{n-1,\alpha},\dots[\omega^1_{2,\alpha},\omega^1_{1,\alpha}]\dots]]$, where $\omega^1_{i,\alpha}\in\Omega^1\mathfrak{a}$. Extend $\hat\pi$ and $\hat\sigma$ recursively to $\Omega^n\mathfrak{a}$ by

$$\begin{aligned}\hat\pi([\omega^1,\omega^k]) &:= \hat\pi(\omega^1)\hat\pi(\omega^k) - (-1)^k\hat\pi(\omega^k)\hat\pi(\omega^1)\;, \\ \hat\sigma([\omega^1,\omega^k]) &:= \hat\sigma(\omega^1)\hat\pi(\omega^k) - \hat\pi(\omega^k)\hat\sigma(\omega^1) - \hat\pi(\omega^1)\hat\sigma(\omega^k) - (-1)^k\hat\sigma(\omega^k)\hat\pi(\omega^1)\;.\end{aligned} \tag{2.3}$$

Define for $n\geq 2$

$$\hat\pi(\mathcal{J}^n\mathfrak{a}) := \{\;\hat\sigma(\omega^{n-1})\;,\;\omega^{n-1}\in\Omega^{n-1}\mathfrak{a}\cap\ker\hat\pi\;\}\;. \tag{2.4}$$

Define spaces $\mathbb{r}^0\mathfrak{a}\subset\mathrm{M}_F\mathbb{C}$ and $\mathbb{r}^1\mathfrak{a}\subset\mathrm{M}_F\mathbb{C}$ elementwise by

$$\begin{array}{ll} \mathbb{r}^0\mathfrak{a} = -(\mathbb{r}^0\mathfrak{a})^* = \hat\Gamma(\mathbb{r}^0\mathfrak{a})\hat\Gamma\;, & \mathbb{r}^1\mathfrak{a} = -(\mathbb{r}^1\mathfrak{a})^* = -\hat\Gamma(\mathbb{r}^1\mathfrak{a})\hat\Gamma\;, \\ {}[\mathbb{r}^0\mathfrak{a},\hat\pi(\mathfrak{a})]\subset\hat\pi(\mathfrak{a})\;, & [\mathbb{r}^0\mathfrak{a},\hat\pi(\Omega^1\mathfrak{a})]\subset\hat\pi(\Omega^1\mathfrak{a})\;, \\ \{\mathbb{r}^0\mathfrak{a},\hat\pi(\mathfrak{a})\}\subset\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}+\hat\pi(\Omega^2\mathfrak{a})\;, & \{\mathbb{r}^0\mathfrak{a},\hat\pi(\Omega^1\mathfrak{a})\}\subset\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^1\mathfrak{a})\}+\hat\pi(\Omega^3\mathfrak{a})\;, \\ {}[\mathbb{r}^1\mathfrak{a},\hat\pi(\mathfrak{a})]\subset\hat\pi(\Omega^1\mathfrak{a})\;, & \{\mathbb{r}^1\mathfrak{a},\hat\pi(\Omega^1\mathfrak{a})\}\subset\hat\pi(\Omega^2\mathfrak{a})+\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}\;. \end{array} \tag{2.5}$$

Define spaces $\mathbb{j}^0\mathfrak{a}, \mathbb{j}^1\mathfrak{a}, \mathbb{j}^2\mathfrak{a} \subset \mathrm{M}_F\mathbb{C}$ elementwise by

$$
\begin{aligned}
&\mathbb{j}^0\mathfrak{a} := \mathbb{c}^0\mathfrak{a}\;, && \mathbb{j}^1\mathfrak{a} := \mathbb{c}^1\mathfrak{a}\;, \\
&\mathbb{j}^2\mathfrak{a} := \mathbb{c}^2\mathfrak{a} + \hat{\pi}(\mathcal{J}^2\mathfrak{a}) + \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}\;, && \text{where} \\
&\mathbb{c}^0\mathfrak{a} = -(\mathbb{c}^0\mathfrak{a})^* = \hat{\Gamma}(\mathbb{c}^0\mathfrak{a})\hat{\Gamma}\;, && \mathbb{c}^1\mathfrak{a} = -(\mathbb{c}^0\mathfrak{a})^* = -\hat{\Gamma}(\mathbb{c}^0\mathfrak{a})\hat{\Gamma}\;, \\
&\mathbb{c}^2\mathfrak{a} = (\mathbb{c}^0\mathfrak{a})^* = \hat{\Gamma}(\mathbb{c}^0\mathfrak{a})\hat{\Gamma}\;, \\
&\mathbb{c}^0\mathfrak{a} \cdot \hat{\pi}(\mathfrak{a}) = 0\;, && \mathbb{c}^0\mathfrak{a} \cdot \hat{\pi}(\Omega^1\mathfrak{a}) = 0\;, \\
&\mathbb{c}^1\mathfrak{a} \cdot \hat{\pi}(\mathfrak{a}) = 0\;, && \mathbb{c}^1\mathfrak{a} \cdot \hat{\pi}(\Omega^1\mathfrak{a}) = 0\;, \\
&[\mathbb{c}^2\mathfrak{a}, \hat{\pi}(\mathfrak{a})] = 0\;, && [\mathbb{c}^2\mathfrak{a}, \hat{\pi}(\Omega^1\mathfrak{a})] = 0\;.
\end{aligned}
\tag{2.6}
$$

The connection form ρ has the structure

$$
\begin{aligned}
&\rho = \textstyle\sum_\alpha (c^1_\alpha \otimes m^0_\alpha + c^0_\alpha \gamma^5 \otimes m^1_\alpha)\;, \\
&\quad c^1_\alpha \in \Lambda^1\;, \quad c^0_\alpha \in \Lambda^0\;, \quad m^0_\alpha \in \mathbb{r}^0\mathfrak{a}\;, \quad m^1_\alpha \in \mathbb{r}^1\mathfrak{a}\;,
\end{aligned}
\tag{2.7}
$$

where Λ^k is the space of differential k–forms represented by gamma matrices. The curvature θ is computed from the connection form ρ by

$$
\begin{aligned}
\theta &= \mathbf{d}\rho + \rho^2 - \mathrm{i}\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}_{\mathfrak{g}}(\rho)\gamma^5 + \mathbb{J}^2\mathfrak{g}\;, \\
\mathbb{J}^2\mathfrak{g} &= (\Lambda^2 \otimes \mathbb{j}^0\mathfrak{a}) \oplus (\Lambda^1\gamma^5 \otimes \mathbb{j}^1\mathfrak{a}) \oplus (\Lambda^0 \otimes \mathbb{j}^2\mathfrak{a})\;,
\end{aligned}
\tag{2.8}
$$

where $\mathbf{d}$ is the exterior differential and $\hat{\sigma}_{\mathfrak{g}}$ the extension of $\mathrm{id} \otimes \hat{\sigma}$ to elements of the form (2.7). Select the representative $\mathfrak{e}(\theta)$ orthogonal to $\mathbb{J}^2\mathfrak{g}$, i.e. find $\mathrm{j} \in \mathbb{J}^2\mathfrak{g}$ such that

$$
\begin{aligned}
&\mathfrak{e}(\theta) = \mathbf{d}\rho + \rho^2 - \mathrm{i}\{\gamma^5 \otimes \mathcal{M}, \rho\} + \hat{\sigma}(\rho)\gamma^5 + \mathrm{j}\;, \\
&\int_X dx\;\; \mathrm{tr}_c(\mathfrak{e}(\theta)\,\mathrm{j}_2) = 0\;, \quad \forall\, \mathrm{j}_2 \in \mathbb{J}^2\mathfrak{g}\;.
\end{aligned}
\tag{2.9}
$$

The trace tr_c includes the trace in $\mathrm{M}_F\mathbb{C}$ and over gamma matrices. Compute the bosonic and fermionic actions

$$
S_B = \int_X dx\; \frac{1}{g_0^2\, F}\; \mathrm{tr}_c(\mathfrak{e}(\theta)^2)\;, \qquad S_F = \int_X dx\; \boldsymbol{\psi}^*(D + \mathrm{i}\rho)\boldsymbol{\psi}\;, \tag{2.10}
$$

where g_0 is a coupling constant and $\boldsymbol{\psi} \in h$. Finally, perform a Wick rotation to Minkowski space.

3. The Matrix Part of the Unification Model

3.1. **The Representations under Consideration.** We shall adapt our notations to the $\mathrm{SU}(5) \times \mathrm{U}(1)$–model. In contrast to what one could expect from the classical treatment [7] of that model, the matrix Lie algebra is not $\mathrm{su}(5) \oplus \mathrm{u}(1)$ but $\mathfrak{a} = \mathrm{su}(5)$. In our approach, the $\mathrm{u}(1)$–part is not an input of the model but an algebraic consequence. The internal Hilbert space is

$$
\mathbb{C}^{192} = \big(\underline{10} \oplus \underline{5}^* \oplus \underline{1} \oplus \underline{10}^* \oplus \underline{5} \oplus \underline{1}\big) \otimes \mathbb{C}^2 \otimes \mathbb{C}^3\;, \tag{3.1}
$$

where $\underline{10}, \underline{10}^*, \underline{5}, \underline{5}^*, \underline{1}$ are representations of $\mathrm{su}(5)$. Since we consider linear operators on $\mathbb{C}^{192}$, we need the decomposition rules for homomorphisms between the

su(5)–representations occurring in (3.1):

$$
\begin{array}{llll}
\mathrm{End}(\underline{10}) & = \mathrm{End}(\underline{10}^*) & = \underline{10} \otimes \underline{10}^* = \underline{1} \oplus \underline{24} \oplus \underline{75} & \text{(3.2a)} \\
\mathrm{End}(\underline{5}) & = \mathrm{End}(\underline{5}^*) & = \underline{5} \otimes \underline{5}^* = \underline{1} \oplus \underline{24} & \text{(3.2b)} \\
\mathrm{End}(\underline{1}) & & = \underline{1} & \text{(3.2c)} \\
\mathrm{Hom}(\underline{5}, \underline{10}) & = \mathrm{Hom}(\underline{10}^*, \underline{5}^*) & = \underline{5}^* \otimes \underline{10} = \underline{5} \oplus \underline{45}^* & \text{(3.2d)} \\
\mathrm{Hom}(\underline{5}, \underline{10}^*) & = \mathrm{Hom}(\underline{10}, \underline{5}^*) & = \underline{5}^* \otimes \underline{10}^* = \underline{10} \oplus \underline{40}^* & \text{(3.2e)} \\
\mathrm{Hom}(\underline{5}^*, \underline{5}) & & = \underline{5} \otimes \underline{5} = \underline{10} \oplus \underline{15} & \text{(3.2f)} \\
\mathrm{Hom}(\underline{10}^*, \underline{10}) & & = \underline{10} \otimes \underline{10} = \underline{5}^* \oplus \underline{45} \oplus \underline{50} & \text{(3.2g)} \\
\mathrm{Hom}(\underline{1}, \underline{5}) & = \mathrm{Hom}(\underline{5}^*, \underline{1}) & = \underline{5} & \text{(3.2h)} \\
\mathrm{Hom}(\underline{1}, \underline{10}) & = \mathrm{Hom}(\underline{10}^*, \underline{1}) & = \underline{10} & \text{(3.2i)}
\end{array}
$$

We identify the matrix Lie algebra su(5) with its $\underline{24}$–representation. Then, we get a natural representation $\hat{\pi}$ of su(5) in $\mathrm{End}(\mathbb{C}^{192})$ by selecting the $\underline{24}$–representations in (3.2):

$$
\hat{\pi}(a) := \left(\begin{array}{ccc|ccc}
\pi_{10}(a) & 0 & 0 & & & \\
0 & \overline{\pi_5(a)} & 0 & & \mathrm{O} & \\
0 & 0 & 0_3 & & & \\
\hline
& & & \overline{\pi_{10}(a)} & 0 & 0 \\
& \mathrm{O} & & 0 & \pi_5(a) & 0 \\
& & & 0 & 0 & 0_3
\end{array}\right) \otimes \mathbb{1}_6 \;. \qquad (3.3)
$$

Here, π_{10} and π_5 denote the embeddings of $\underline{24}$ into (3.2).

We define the $\underline{75}$–representation of su(5) occurring in the decomposition (3.2a) as the set $\mathfrak{v}$ of 10×10–matrices of the form

$$
\mathfrak{v} := \{\; v \in \mathrm{su}(10) \;,\; \mathrm{tr}(v\, \pi_{10}(a)) = 0 \;\forall a \in \mathfrak{a} \;\} \;. \qquad (3.4)
$$

Next, we consider the $\underline{5}$–representations occurring on the r.h.s. of (3.2). Let $\mathfrak{b} = \mathbb{C}^5$ be the vector space of matrices represented in the form

$$
b = \mathrm{i}(b_1, b_2, b_3, b_4, b_5)^T \;, \quad b_i \in \mathbb{C} \;. \qquad (3.5)
$$

We define a linear map $\hat{\pi}$ of $\mathfrak{b}$ in $\mathrm{End}(\mathbb{C}^{192})$, putting

$$
\hat{\pi}(b) = \left(\begin{array}{ccc|ccc}
& & & \pi_{10,10}(b) & \pi_{10,5}(b) & 0 \\
& \mathrm{O} & & \pi_{10,5}(b)^T & 0 & \pi_{5,1}(b) \\
& & & 0 & \pi_{5,1}(b)^T & 0 \\
\hline
-\pi_{10,10}(b)^* & -\overline{\pi_{10,5}(b)} & 0 & & & \\
-\pi_{10,5}(b)^* & 0 & -\overline{\pi_{5,1}(b)} & & \mathrm{O} & \\
0 & -\pi_{5,1}(b)^* & 0 & & &
\end{array}\right) \otimes \mathbb{1}_6 \;. \qquad (3.6)
$$

The matrices $\pi_{10,10}(b), \pi_{10,5}(b)$ and $\pi_{5,1}(b)$ are the embeddings of $b \in \underline{5}$ into $\underline{10} \otimes \underline{10}$, $\underline{5}^* \otimes \underline{10}$ and $\underline{1} \otimes \underline{5}^*$, see (3.2). Observe that

$$
[\hat{\pi}(a), \hat{\pi}(b)] = \hat{\pi}(ab) \in \hat{\pi}(\mathfrak{b}) \;, \quad a \in \mathfrak{a} \;,\; b \in \mathfrak{b} \;. \qquad (3.7)
$$

Due to the first three formulae in (3.2), the $\underline{24}$–parts and the $\underline{1}$–parts of $\pi_{i,j}(b)\pi_{i,j}(b)^*$, respectively, must be correlated. Indeed, we find with

$$(b,b)' := bb^* - \tfrac{1}{5}\,\mathrm{tr}(bb^*)\mathbb{1}_5 \in \mathrm{i}\mathfrak{a} \tag{3.8}$$

the identities [12]

$$\begin{aligned}
\pi_{10,10}(b)\pi_{10,10}(b)^* &= \mathrm{i}\pi_{10}(\mathrm{i}(b,b)') + \tfrac{3}{5}(b^*b)\mathbb{1}_{10}\ , \\
\pi_{10,5}(b)\pi_{10,5}(b)^* &= -\mathrm{i}\pi_{10}(\mathrm{i}(b,b)') + \tfrac{2}{5}(b^*b)\mathbb{1}_{10}\ , \\
\pi_{10,5}(b)^*\pi_{10,5}(b) &= \mathrm{i}\pi_{5}(\mathrm{i}(b,b)') + \tfrac{4}{5}(b^*b)\mathbb{1}_{5}\ , \\
\overline{\pi_{5,1}(b)}\pi_{5,1}(b)^T &= -\mathrm{i}\pi_{5}(\mathrm{i}(b,b)') + \tfrac{1}{5}(b^*b)\mathbb{1}_{5}\ , \\
\pi_{5,1}(b)^T\overline{\pi_{5,1}(b)} &= (b^*b)\ .
\end{aligned} \tag{3.9}$$

Moreover, we consider the $\underline{45}$–representation of su(5) occurring in (3.2d). It is the vector space $\mathfrak{w}$ of 10×5–matrices determined by

$$\mathfrak{w} := \{\ w \in \mathrm{Hom}(\mathbb{C}^5,\mathbb{C}^{10})\ ,\ \mathrm{tr}(w\,\pi_{10,5}(b)^*) = 0\ \forall b\in\underline{5}\ \}\ . \tag{3.10a}$$

One has

$$[a,w] := \pi_{10}(a)w - w\pi_5(a) \in \mathfrak{w}\ , \quad w\in\mathfrak{w}\,,\ a\in\mathfrak{a}\,. \tag{3.10b}$$

Finally, we consider the $\underline{50}$–representation of su(5) occurring in (3.2g). It is the vector space $\mathfrak{c}$ of symmetric complex 10×10–matrices determined by

$$\mathfrak{c} := \{\ c\in \mathrm{M}_{10}\mathbb{C}\ ,\ c = c^T\ ,\ \mathrm{tr}(c\,\pi_{10,10}(b)^*) = 0\ \forall b\in\underline{5}\ \}\ . \tag{3.11a}$$

One has

$$[a,c] := \pi_{10}(a)c - c\overline{\pi_{10}(a)} \equiv \pi_{10}(a)c + c\pi_{10}(a)^T \in \mathfrak{c}\ , \quad a\in\mathfrak{a}\,,\ c\in\mathfrak{c}\ . \tag{3.11b}$$

3.2. **The Mass Matrix.** Now we define the mass matrix $\mathcal{M}$ of the L–cycle. Let

$$\begin{aligned}
&m \equiv \pi_5(m) := \mathrm{i}\ \mathrm{diag}(-\tfrac{2}{5},-\tfrac{2}{5},-\tfrac{2}{5},\tfrac{3}{5},\tfrac{3}{5}) \in \mathfrak{a}\ , \\
&\qquad \pi_{10}(m) \equiv \mathrm{i}\ \mathrm{diag}(\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{5},-\tfrac{4}{5},-\tfrac{4}{5},-\tfrac{4}{5},\tfrac{6}{5})\ , \\
&n := \mathrm{i}(0,0,0,1,0)^T \in \mathfrak{b}\ , \\
&m' := \mathrm{i}\begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & -1 \end{pmatrix} \in \mathfrak{c}\ , \quad n' := \mathrm{i}\begin{pmatrix} \mathbb{1}_3 & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times1} & 3 \end{pmatrix} \in \mathfrak{w}\ .
\end{aligned} \tag{3.12}$$

Then we put

$$\mathcal{M} = \left(\begin{array}{ccc|ccc}
\mathcal{M}_{10} & 0 & 0 & \mathcal{M}_{10,10} & \mathcal{M}_{10,5} & 0 \\
0 & \overline{\mathcal{M}_5} & 0 & \mathcal{M}_{10,5}^T & 0 & \mathcal{M}_{5,1} \\
0 & 0 & 0 & 0 & \mathcal{M}_{5,1}^T & 0 \\ \hline
\mathcal{M}_{10,10}^* & \overline{\mathcal{M}_{10,5}} & 0 & \overline{\mathcal{M}_{10}} & 0 & 0 \\
\mathcal{M}_{10,5}^* & 0 & \overline{\mathcal{M}_{5,1}} & 0 & \mathcal{M}_5 & 0 \\
0 & \mathcal{M}_{5,1}^* & 0 & 0 & 0 & 0
\end{array}\right)\ , \text{ where} \tag{3.13a}$$

$$\begin{aligned} \mathcal{M}_{10} &= \mathrm{i}\pi_{10}(m)\otimes M'_{10}\;, & \mathcal{M}_{5} &= -\mathrm{i}\pi_{5}(m)\otimes M'_{5}\;, \\ \mathcal{M}_{10,10} &= \mathrm{i}\pi_{10,10}(n)\otimes M'_{d} + \mathrm{i}m'\otimes M'_{N}\;, & \mathcal{M}_{5,1} &= \mathrm{i}\pi_{5,1}(n)\otimes M'_{e}\;, \\ \mathcal{M}_{10,5} &= \mathrm{i}\pi_{10,5}(n)\otimes M'_{\tilde u} + \mathrm{i}n'\otimes M'_{\tilde n}\;. \end{aligned} \tag{3.13b}$$

Here, $M'_{10}, M'_5, M'_N, M'_{\tilde u}, M'_d, M'_e, M'_{\tilde n}$ are 6×6–matrices of the following block structure:

$$M'_i = \begin{pmatrix} 0_3 & M_i \\ M_i^* & 0_3 \end{pmatrix}\;, \qquad M'_f = \begin{pmatrix} M_f & 0_3 \\ 0_3 & M_f \end{pmatrix}\;, \tag{3.14}$$

for $i \in \{5,10\}$ and $f \in \{\tilde u, d, e, \tilde n, N\}$. The only condition to the 3×3–mass matrices $M_{10}, M_5, M_{\tilde u}, M_d, M_e, M_{\tilde n}$ and M_N is

$$M_d = M_d^T\;, \qquad\qquad M_N = M_N^T\;. \tag{3.15}$$

The final input of our L–cycle is the grading operator $\hat\Gamma$, which we choose as

$$\hat\Gamma = \begin{pmatrix} -\mathbb{1}_{16}\otimes\hat\Gamma' & 0_{96} \\ 0_{96} & \mathbb{1}_{16}\otimes\hat\Gamma' \end{pmatrix}\;, \qquad \hat\Gamma' = \begin{pmatrix} \mathbb{1}_3 & 0 \\ 0 & -\mathbb{1}_3 \end{pmatrix}\;. \tag{3.16}$$

From (3.3) and (3.16) there follows that $\hat\Gamma$ commutes with $\hat\pi(\mathfrak{a})$. The fact that $\hat\Gamma'$ commutes with $M'_{\tilde u,d,e,\tilde n,N}$ and anticommutes with $M'_{10,5}$ implies that $\hat\Gamma$ anticommutes with $\mathcal{M}$. Therefore, the tuple $(\mathfrak{a}, \mathbb{C}^{192}, \mathcal{M}, \hat\pi, \hat\Gamma)$ is an L–cycle.

Let us summarize the block structure of this L–cycle, for instance in terms of 4×4–block matrices with entries in 48×48–matrices:

$$\hat\pi(a) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & \bar A & 0 \\ 0 & 0 & 0 & \bar A \end{pmatrix},\; \mathcal{M} = \begin{pmatrix} 0 & \mathcal{M}_i & \mathcal{M}_f & 0 \\ \mathcal{M}_i^* & 0 & 0 & \mathcal{M}_f \\ \mathcal{M}_f^* & 0 & 0 & \overline{\mathcal{M}}_i \\ 0 & \mathcal{M}_f^* & \mathcal{M}_i^T & 0 \end{pmatrix},\; \hat\Gamma = \begin{pmatrix} -\mathbb{1}_{48} & 0 & 0 & 0 \\ 0 & \mathbb{1}_{48} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{48} & 0 \\ 0 & 0 & 0 & -\mathbb{1}_{48} \end{pmatrix},$$

with (3.17)

$$\begin{aligned} A &:= \mathrm{diag}\left(\pi_{10}(a)\otimes\mathbb{1}_3\;,\;\overline{\pi_5(a)}\otimes\mathbb{1}_3\;,\;0_3\right)\;, \\ \mathcal{M}_i &:= \mathrm{diag}\left(\mathrm{i}\pi_{10}(m)\otimes M_{10}\;,\;\overline{-\mathrm{i}\pi_5(m)\otimes M_5}\;,\;0_3\right)\;, \\ \mathcal{M}_f &:= \begin{pmatrix} \mathrm{i}\pi_{10,10}(n)\otimes M_d + \mathrm{i}m'\otimes M_N & \mathrm{i}\pi_{10,5}(n)\otimes M_{\tilde u} + \mathrm{i}n'\otimes M_{\tilde n} & 0 \\ \mathrm{i}\pi_{10,5}(n)^T\otimes M_{\tilde u}^T + \mathrm{i}n'^T\otimes M_{\tilde n}^T & 0 & \mathrm{i}\pi_{5,1}(n)\otimes M_e \\ 0 & \mathrm{i}\pi_{5,1}(n)^T\otimes M_e^T & 0 \end{pmatrix} \equiv \mathcal{M}_f^T\;. \end{aligned}$$

3.3. **The Structure of $\hat\pi(\Omega^1\mathfrak{a})$ and $\hat\pi(\Omega^2\mathfrak{a})$.** We recall (2.2a) that elements $\tau^1 \in \hat\pi(\Omega^1\mathfrak{a})$ are of the form

$$\tau^1 = \textstyle\sum_{\alpha,z\geq 0}[\hat\pi(a_\alpha^z), [\dots[\hat\pi(a_\alpha^1), [-\mathrm{i}\mathcal{M}, \hat\pi(a_\alpha^0)]]\dots]]\;. \tag{3.18}$$

Using (3.7), (3.10b) and the fact that $\hat\pi$ is a representation we obtain the explicit structure of elements $\tau^1 \in \hat\pi(\Omega^1\mathfrak{a})$:

$$\tau^1 = \tag{3.19a}$$

$$\left(\begin{array}{ccc|ccc}
\pi_{10}(a)\otimes M'_{10} & 0 & 0 & \begin{bmatrix}\pi_{10,10}(b)\otimes M'_d \\ +c\otimes M'_N\end{bmatrix} & \begin{bmatrix}\pi_{10,5}(b)\otimes M'_{\tilde{u}} \\ +w\otimes M'_{\tilde{n}}\end{bmatrix} & 0 \\
0 & \overline{\pi_5(a)\otimes M'_5} & 0 & \begin{bmatrix}\pi_{10,5}(b)^T\otimes M'^{\,T}_{\tilde{u}} \\ +w^T\otimes M'^{\,T}_{\tilde{n}}\end{bmatrix} & 0 & \pi_{5,1}(b)\otimes M'_e \\
0 & 0 & 0 & 0 & \pi_{5,1}(b)^T\otimes M'^{\,T}_e & 0 \\
\hline
\begin{bmatrix}-\pi_{10,10}(b)^*\otimes M'^{\,*}_d \\ -c^*\otimes M'_N\end{bmatrix} & \begin{bmatrix}-\overline{\pi_{10,5}(b)\otimes M'_{\tilde{u}}} \\ -\overline{w\otimes M'_{\tilde{n}}}\end{bmatrix} & 0 & -\overline{\pi_{10}(a)\otimes M'_{10}} & 0 & 0 \\
\begin{bmatrix}-\pi_{10,5}(b)^*\otimes M'^{\,*}_{\tilde{u}} \\ -w^*\otimes M'^{\,*}_{\tilde{n}}\end{bmatrix} & 0 & -\overline{\pi_{5,1}(b)\otimes M'_e} & 0 & -\pi_5(a)\otimes M'_5 & 0 \\
0 & -\pi_{5,1}(b)^*\otimes M'^{\,*}_e & 0 & 0 & 0 & 0
\end{array}\right),$$

$$a = \textstyle\sum_{\alpha,z\geq 0}[a^z_\alpha,[\dots[a^1_\alpha,[m,a^0_\alpha]]\dots]] \in \mathfrak{a}\ , \tag{3.19b}$$

$$b = -\textstyle\sum_{\alpha,z\geq 0} a^z_\alpha\cdots a^1_\alpha a^0_\alpha n \in \mathfrak{b}\ , \tag{3.19c}$$

$$w = \textstyle\sum_{\alpha,z\geq 0}[a^z_\alpha,[\dots[a^1_\alpha,[n',a^0_\alpha]]\dots]] \in \mathfrak{w}\ , \tag{3.19d}$$

$$c = \textstyle\sum_{\alpha,z\geq 0}[a^z_\alpha,[\dots[a^1_\alpha,[m',a^0_\alpha]]\dots]] \in \mathfrak{c}\ . \tag{3.19e}$$

Here, the commutators (3.19d) and (3.19e) are understood in the sense (3.10b) and (3.11b). It is obvious that a,b,c,w are independent as elements of different irreducible representations of su(5).

Next, we are going to construct $\hat{\pi}(\Omega^2\mathfrak{a})$. According to (2.3), elements $\tau^2 \in \hat{\pi}(\Omega^2\mathfrak{a})$ are obtained by summing up elements of the type

$$\tau^2 := \tfrac{1}{2}\{\tau^1,\tau^1\}\ , \qquad \tau^1 \in \hat{\pi}(\Omega^1\mathfrak{a})\ . \tag{3.20}$$

Thus, using (3.9) we get from (3.19a) the structure

$$\tau^2 = \left(\begin{array}{ccc|ccc}
\tau_{10} & \tau_{\widetilde{10,5}} & \tau_{10,1} & \tau_{10,10} & \tau_{10,5} & 0 \\
\tau^*_{\widetilde{10,5}} & \tau^T_5 & 0 & \tau^T_{10,5} & 0 & \tau_{5,1} \\
\tau^*_{10,1} & 0 & \tau_1 & 0 & \tau^T_{5,1} & 0 \\
\hline
\tau^*_{10,10} & \overline{\tau_{10,5}} & 0 & \tau^T_{10} & \overline{\tau_{\widetilde{10,5}}} & \overline{\tau_{10,1}} \\
\tau^*_{10,5} & 0 & \overline{\tau_{5,1}} & \tau^T_{\widetilde{10,5}} & \tau_5 & 0 \\
0 & \tau^*_{5,1} & 0 & \tau^T_{10,1} & 0 & \tau^T_1
\end{array}\right)\ , \text{ where} \tag{3.21a}$$

$$\begin{aligned}
\tau_{10} = {} & \mathrm{i}\pi_{10}(\mathrm{i}(b,b)')\otimes(M'_{\tilde{u}}M'^{\,*}_{\tilde{u}} - M'_dM'^{\,*}_d) - (b^*b)\mathbb{1}_{10}\otimes(\tfrac{2}{5}M'_{\tilde{u}}M'^{\,*}_{\tilde{u}} + \tfrac{3}{5}M'_dM'^{\,*}_d) \\
& +\tfrac{1}{2}\{\pi_{10}(a),\pi_{10}(a)\}\otimes M'^{\,2}_{10} \\
& -ww^*\otimes M'_{\tilde{n}}M'^{\,*}_{\tilde{n}} - w\pi_{10,5}(b)^*\otimes M'_{\tilde{n}}M'^{\,*}_{\tilde{u}} - \pi_{10,5}(b)w^*\otimes M'_{\tilde{u}}M'^{\,*}_{\tilde{n}} \\
& -cc^*\otimes M'_NM'^{\,*}_N - c\pi_{10,10}(b)^*\otimes M'_NM'^{\,*}_d - \pi_{10,10}(b)c^*\otimes M'_dM'^{\,*}_N \\
\tau_5 = {} & \mathrm{i}\pi_5(\mathrm{i}(b,b)')\otimes(\bar{M}'_eM'^{\,T}_e - M'^{\,*}_{\tilde{u}}M'_{\tilde{u}}) - (b^*b)\mathbb{1}_5\otimes(\tfrac{4}{5}M'^{\,*}_{\tilde{u}}M'_{\tilde{u}} + \tfrac{1}{5}\bar{M}'_eM'^{\,T}_e) \\
& +\tfrac{1}{2}\{\pi_5(a),\pi_5(a)\}\otimes M'^{\,2}_5 \qquad\qquad (3.21\text{b}) \\
& -w^*w\otimes M'^{\,*}_{\tilde{n}}M'_{\tilde{n}} - w^*\pi_{10,5}(b)\otimes M'^{\,*}_{\tilde{n}}M'_{\tilde{u}} - \pi_{10,5}(b)^*w\otimes M'^{\,*}_uM'_{\tilde{n}}\ , \\
\tau_1 = {} & -b^*b\otimes M'^{\,T}_e\bar{M}'_e\ ,
\end{aligned}$$

$$
\begin{aligned}
\tau_{10,10} &= \pi_{10,10}(ab) \otimes \tfrac{1}{2}(M'_{10}M'_d + M'_d M'^T_{10}) \\
&\quad +(\pi_{10}(a)\pi_{10,10}(b) - \pi_{10,10}(b)\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_d - M'_d M'^T_{10}) \\
&\quad +(\pi_{10}(a)c + c\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N M'^T_{10}) \\
&\quad +(\pi_{10}(a)c - c\pi_{10}(a)^T) \otimes \tfrac{1}{2}(M'_{10}M'_N - M'_N M'^T_{10})\ , \\
\tau_{10,5} &= \pi_{10}(a)\pi_{10,5}(b) \otimes M'_{10}M'_{\tilde u} - \pi_{10,5}(b)\pi_5(a) \otimes M'_{\tilde u}M'_5 \\
&\quad +\pi_{10}(a)w \otimes M'_{10}M'_{\tilde n} - w\pi_5(a) \otimes M'_{\tilde n}M'_5\ , \\
\tau_{\widetilde{10},\tilde 5} &= -\pi_{10,10}(b)\overline{w} \otimes M'_d \bar M'_{\tilde n} - c\overline{w} \otimes M'_N \bar M'_{\tilde n} - \overline{c\pi_{10,5}(b)} \otimes M'_N \bar M'_{\tilde u}\ , \\
\tau_{5,1} &= \pi_{5,1}(ab) \otimes M'^T_5 M'_e\ , \qquad \tau_{10,1} = -\overline{w\pi_{5,1}(b)} \otimes M'_{\tilde n}\bar M'_e\ .
\end{aligned}
\tag{3.21c}
$$

3.4. **The Structure of the Connection Form.** We know from (2.7) that for constructing the connection form ρ we need knowledge of the spaces $\mathfrak{r}^0\mathfrak{a}$ and $\mathfrak{r}^1\mathfrak{a}$ determined by the equations (2.5). To compute the structure of elements $\eta^0 \in \mathfrak{r}^0\mathfrak{a}$ we first decompose η^0 according to (3.2) into irreducible su(5)–representations, each of them tensorized by $\mathrm{M}_6\mathbb{C}$. Then, the condition $[\mathfrak{r}^0\mathfrak{a}, \hat\pi(\mathfrak{a})] \subset \hat\pi(\mathfrak{a})$ yields the block structure

$$
\eta^0 = \hat\pi(a) + \mathrm{i}\ \mathrm{diag}(\mathbb{1}_{10} \otimes m'_{10}\ ,\ \mathbb{1}_5 \otimes m'_{\tilde 5}\ ,\ m'_1\ ,\ \mathbb{1}_{10} \otimes m'_{\widetilde{10}}\ ,\ \mathbb{1}_5 \otimes m'_5\ ,\ m'_{\tilde 1})\ ,
$$

where $a \in \mathfrak{a}$ and $m'_{10,\tilde 5,1,\widetilde{10},5,\tilde 1}$ are selfadjoint elements of $\mathrm{M}_6\mathbb{C}$. The condition $\mathfrak{r}^0\mathfrak{a} = \hat\Gamma(\mathfrak{r}^0\mathfrak{a})\hat\Gamma$ implies $m'_i = \mathrm{diag}(m_i, \hat m_i)$, for $m_i, \hat m_i \in \mathrm{M}_3\mathbb{C}$.

We insert this structure into the condition $[\mathfrak{r}^0\mathfrak{a}, \hat\pi(\Omega^1\mathfrak{a})] \subset \hat\pi(\Omega^1\mathfrak{a})$. Using (3.19a), (3.7), (3.10b) and (3.11b) we obtain from the off–diagonal blocks the equations

$$
\begin{aligned}
m_{10}M_d - M_d m_{\widetilde{10}} &= -\mathrm{i}\bar\alpha M_d\ , & m_{10}M_N - M_N m_{\widetilde{10}} &= -\mathrm{i}\bar\alpha' M_N\ , \\
m_{10}M_{\tilde u} - M_{\tilde u}m_5 &= -\mathrm{i}\alpha M_{\tilde u}\ , & m_{10}M_{\tilde n} - M_{\tilde n}m_5 &= -\mathrm{i}\alpha'' M_{\tilde n}\ , \\
m_{\tilde 5}M^T_{\tilde u} - M^T_{\tilde u}m_{\widetilde{10}} &= -\mathrm{i}\alpha M^T_{\tilde u}\ , & m_{\tilde 5}M^T_{\tilde n} - M^T_{\tilde n}m_{\widetilde{10}} &= -\mathrm{i}\alpha'' M^T_{\tilde n}\ , \\
m_{\tilde 5}M_e - M_e m_{\tilde 1} &= -\mathrm{i}\bar\alpha M_e\ , & m_1 M^T_e - M^T_e m_5 &= -\mathrm{i}\bar\alpha M^T_e\ ,
\end{aligned}
\tag{3.22a}
$$

for $\alpha, \alpha', \alpha'' \in \mathbb{C}$. The same equations hold for $\hat m_i$, with the same parameters $\alpha, \alpha', \alpha''$. Multiplying the first equation by M^*_d from the right and subtracting the Hermitian conjugate of the resulting equation we get for instance

$$
[m_{10}, M_d M^*_d] = -\mathrm{i}(\alpha + \bar\alpha) M_d M^*_d\ .
$$

Applying the trace and respecting $\mathrm{tr}(M_d M^*_d) > 0$ we get $\alpha = \mathrm{i}\lambda$, for $\lambda \in \mathbb{R}$. Analogously, we have $\alpha' = \mathrm{i}\lambda'$ and $\alpha'' = \mathrm{i}\lambda''$. Thus, we find the equations

$$
[m_{10}, M_d M^*_d] = [m_{10}, M_N M^*_N] = [m_{10}, M_{\tilde u} M^*_{\tilde u}] = [m_{10}, M_{\tilde n} M^*_{\tilde n}] = 0\ . \tag{3.22b}
$$

For generic mass matrices $M_{d,N,\tilde u,\tilde n}$, these equations can only be satisfied for $m_{10} = (\nu - \frac{1}{2}\lambda)\mathbb{1}_3$, for $\nu \in \mathbb{R}$. We assume that $M_{d,\tilde u,e}$ are invertible and find the solution

$$
\begin{aligned}
m_{10} &= (\nu - \tfrac{1}{2}\lambda)\mathbb{1}_3\ , & m_5 &= (\nu - \tfrac{3}{2}\lambda)\mathbb{1}_3\ , & m_1 &= (\nu - \tfrac{5}{2}\lambda)\mathbb{1}_3\ , \\
m_{\widetilde{10}} &= (\nu + \tfrac{1}{2}\lambda)\mathbb{1}_3\ , & m_{\tilde 5} &= (\nu + \tfrac{3}{2}\lambda)\mathbb{1}_3\ , & m_{\tilde 1} &= (\nu + \tfrac{5}{2}\lambda)\mathbb{1}_3\ ,
\end{aligned}
\tag{3.22c}
$$

where $\nu,\lambda\in\mathbb{R}$. For $\hat{m}_i$ we get the same equations, with the same λ but possibly a different $\hat{\nu}$ instead of ν. Inserting this result into the π_{10}–block we get the equations

$$(\nu-\hat{\nu})M_{10}=\beta M_{10}\;, \qquad\qquad (\nu-\hat{\nu})M_{10}^*=-\beta M_{10}^*\;,$$

which are only compatible with $\nu=\hat{\nu}$. Thus, we obtain with (3.3) the preliminary result

$$\begin{aligned} \eta^0 &= \hat{\pi}(a)+\hat{\pi}(\mathrm{u}(1))+\mathrm{i}\nu\mathbb{1}_{192}\;, && (3.23a)\\ \hat{\pi}(\mathrm{i}\lambda) &:= \mathrm{i}\lambda\;\mathrm{diag}(\,-\tfrac{1}{2}\mathbb{1}_{10}\;,\;\tfrac{3}{2}\mathbb{1}_5\;,\;-\tfrac{5}{2}\;,\;\tfrac{1}{2}\mathbb{1}_{10}\;,\;-\tfrac{3}{2}\mathbb{1}_5\;,\;\tfrac{5}{2}\,)\otimes\mathbb{1}_6\;. && (3.23b)\end{aligned}$$

Now, one finds [12] that the u(1)–part $\hat{\pi}(\mathrm{u}(1))$ is compatible with the two conditions $\{\mathrm{r}^0\mathfrak{a},\hat{\pi}(\mathfrak{a})\}\subset\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a})\}+\hat{\pi}(\Omega^2\mathfrak{a})$ and $\{\mathrm{r}^0\mathfrak{a},\hat{\pi}(\Omega^1\mathfrak{a})\}\subset\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^1\mathfrak{a})\}+\hat{\pi}(\Omega^3\mathfrak{a})$, whereas the identity part $\mathrm{i}\nu\mathbb{1}_{192}$ is not. Here, one has to use the following identities:

$$\left.\begin{aligned} \mathrm{tr}(\pi_{10}(a)\,\pi_{10}(a)) &= \mathrm{tr}(\overline{\pi_{10}(a)}\,\overline{\pi_{10}(a)}) = 3\;\mathrm{tr}(aa)\;,\\ \mathrm{tr}(\pi_5(a)\,\pi_5(a)) &= \mathrm{tr}(\overline{\pi_5(a)}\,\overline{\pi_5(a)}) \;\;=\;\; \mathrm{tr}(aa)\;, \end{aligned}\right. \qquad (3.24a)$$

$$\mathrm{i}\{\pi_{10}(a),\pi_{10}(a)\}_{\underline{24}}=\tfrac{1}{3}\pi_{10}\big(\mathrm{i}\{\pi_5(a),\pi_5(a)\}_{\underline{24}}\big)\;, \qquad (3.24b)$$

$$\left.\begin{aligned} (\pi_{10}(a)\pi_{10,5}(b))_{\underline{5}} &= \tfrac{3}{4}\pi_{10,5}(ab)\;, & (\pi_{10,5}(b)\pi_5(a))_{\underline{5}} &= -\tfrac{1}{4}\pi_{10,5}(ab)\;,\\ (\pi_{10}(a)\pi_{10,5}(b))_{\underline{45}} &= (\pi_{10,5}(b)\pi_5(a))_{\underline{45}}\;, & (\pi_{10}(a)w)_{\underline{5}} &= (w\pi_5(a))_{\underline{5}}\;, \end{aligned}\right. \qquad (3.24c)$$

for $a\in\mathfrak{a}$, $b\in\mathfrak{b}$ and $w\in\mathfrak{w}$.

The evaluation of the formulae for $\mathrm{r}^1\mathfrak{a}$ in (2.5) yields for a generic choice of the mass matrices $M_{\tilde{u},d,e,\tilde{n},10,5}$ the simple result $\mathrm{r}^1\mathfrak{a}=\hat{\pi}(\Omega^1\mathfrak{a})$. Therefore, the connection form has the structure

$$\rho\in\big(\Lambda^1\otimes(\hat{\pi}(\mathfrak{a})+\hat{\pi}(\mathrm{u}(1)))\big)\oplus\big(\Lambda^1\gamma^5\otimes\hat{\pi}(\Omega^1\mathfrak{a})\big)\;. \qquad (3.25)$$

We see that our formalism generates an additional u(1)–part for the connection form and determines uniquely its representation (3.23b) on the fermionic Hilbert space. Remarkably, this representation is realized in nature!

3.5. **The Ideal $\mathrm{j}^2\mathfrak{a}$.** We recall (2.4) that for the analysis of $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$ we must find the space of elements $\hat{\sigma}(\omega^1)$, where $\omega^1\in\Omega^1\mathfrak{a}\cap\ker\hat{\pi}$. For the computation of $\hat{\sigma}(\omega^1)$ we need knowledge of $\mathcal{M}^2$, see (2.2b). We define

$$\left.\begin{aligned} v_0 &:= \mathrm{i}\;\mathrm{diag}(-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},1\,)\in\mathfrak{v}\;,\\ I_3\equiv\pi_5(I_3) &:= \mathrm{i}\;\mathrm{diag}\,(0,0,0,\tfrac{1}{2},-\tfrac{1}{2})\;,\\ \pi_{10}(I_3) &\equiv \mathrm{i}\;\mathrm{diag}\,(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},0,0,0,0) \end{aligned}\right. \qquad (3.26)$$

and abbreviate

$$M_u':=M_{\tilde{u}}'+M_{\tilde{n}}'\;, \qquad\qquad M_n':=M_{\tilde{u}}'-3M_{\tilde{n}}'\;, \qquad (3.27)$$

analogously for the primeless matrices $M_{u,\nu,\tilde{u},\tilde{n}}$. Then, using (3.12) and (3.13), we find the following formula for $\mathcal{M}^2$:

$$\mathcal{M}^2 = \left(\begin{array}{ccc|ccc} (\mathcal{M}^2)_{10} & (\mathcal{M}^2)_{\widetilde{10,5}} & 0 & (\mathcal{M}^2)_{10,10} & (\mathcal{M}^2)_{10,5} & 0 \\ (\mathcal{M}^2)^*_{\widetilde{10,5}} & (\mathcal{M}^2)^T_5 & 0 & (\mathcal{M}^2)^T_{10,5} & 0 & (\mathcal{M}^2)_{5,1} \\ 0 & 0 & (\mathcal{M}^2)_1 & 0 & (\mathcal{M}^2)^T_{5,1} & 0 \\ \hline (\mathcal{M}^2)^*_{10,10} & \overline{(\mathcal{M}^2)_{10,5}} & 0 & (\mathcal{M}^2)^T_{10} & \overline{(\mathcal{M}^2)_{\widetilde{10,5}}} & 0 \\ (\mathcal{M}^2)^*_{10,5} & 0 & \overline{(\mathcal{M}^2)_{5,1}} & (\mathcal{M}^2)^T_{\widetilde{10,5}} & (\mathcal{M}^2)_5 & 0 \\ 0 & (\mathcal{M}^2)^*_{5,1} & 0 & 0 & 0 & (\mathcal{M}^2)^T_1 \end{array}\right) , \qquad (3.28a)$$

where

$$\begin{aligned}
(\mathcal{M}^2)_{10} = {} & \mathbb{1}_{10} \otimes \big(\tfrac{9}{25}{M'_{10}}^2 + \tfrac{4}{10}M'_{\tilde{u}}{M'_{\tilde{u}}}^* + \tfrac{6}{10}M'_d{M'_d}^* + \tfrac{12}{10}M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \tfrac{1}{10}M'_N{M'_N}^*\big) \\
& -\mathrm{i}v_0 \otimes \big({M'_{10}}^2 - 2(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + 4M'_{\tilde{n}}{M'_{\tilde{n}}}^* + \tfrac{1}{2}M'_N{M'_N}^*\big) \\
& -\mathrm{i}\pi_{10}(\tfrac{Y'}{2} + I_3) \otimes (M'_u{M'_u}^* - M'_d{M'_d}^*) \\
& -\tfrac{1}{3}\mathrm{i}\pi_{10}(m)\big(\tfrac{1}{5}{M'_{10}}^2 - 4(M'_{\tilde{u}}{M'_{\tilde{n}}}^* + M'_{\tilde{n}}{M'_{\tilde{u}}}^*) + 8M'_{\tilde{n}}{M'_{\tilde{n}}}^* + M'_N{M'_N}^*\big) , \\
(\mathcal{M}^2)_5 = {} & \mathbb{1}_5 \otimes \big(\tfrac{6}{25}{M'_5}^2 + \tfrac{12}{5}{M'_{\tilde{n}}}^*M'_{\tilde{n}} + \tfrac{4}{5}{M'_{\tilde{u}}}^*M'_{\tilde{u}} + \tfrac{1}{5}\bar{M}'_e{M'_e}^T\big) \\
& - \mathrm{i}\pi_5(\tfrac{Y'}{2} + I_3) \otimes (\bar{M}'_e{M'_e}^T - {M'_n}^*M'_n) \qquad (3.28b) \\
& -\mathrm{i}\pi_5(m)\big(\tfrac{1}{5}{M'_5}^2 - 4({M'_{\tilde{u}}}^*M'_{\tilde{n}} + {M'_{\tilde{n}}}^*M'_{\tilde{u}}) + 8{M'_{\tilde{n}}}^*M'_{\tilde{n}}\big) , \\
(\mathcal{M}^2)_1 = {} & {M'_e}^T\bar{M}'_e , \\
(\mathcal{M}^2)_{10,10} = {} & \tfrac{3\mathrm{i}}{5}\pi_{10,10}(n) \otimes \tfrac{1}{2}(M'_{10}M'_d + M_d{M'_{10}}^T) \\
& +\tfrac{\mathrm{i}}{2}\pi_{10,10}(n') \otimes \tfrac{1}{2}(M'_{10}M'_d - M_d{M'_{10}}^T) - \tfrac{12\mathrm{i}}{5}m' \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N\overline{M'_{10}}) , \\
(\mathcal{M}^2)_{5,1} = {} & \tfrac{3\mathrm{i}}{5}\pi_{5,1}(n) \otimes {M'_5}^TM'_e , \\
(\mathcal{M}^2)_{10,5} = {} & -\mathrm{i}\pi_{10,5}(n) \otimes \big(\tfrac{9}{20}M'_{10}M'_{\tilde{u}} + \tfrac{3}{20}M'_{\tilde{u}}M'_5 - \tfrac{3}{4}M'_{10}M'_{\tilde{n}} + \tfrac{3}{4}M'_{\tilde{n}}M'_5\big) \\
& -\mathrm{i}n' \otimes \big(-\tfrac{1}{4}M'_{10}M'_{\tilde{u}} + \tfrac{1}{4}M'_{\tilde{u}}M'_5 + \tfrac{19}{20}M'_{10}M'_{\tilde{n}} - \tfrac{7}{20}M'_{\tilde{n}}M'_5\big) , \\
(\mathcal{M}^2)_{\widetilde{10,5}} = {} & -\mathrm{i}n'' \otimes M'_N\overline{M'_n} .
\end{aligned}$$

Here, n'' is a generator of the $\underline{40}^*$–representation of su(5) occurring in the decomposition (3.2e):

$$n'' := \mathrm{i}\begin{pmatrix} 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times1} & 1 \end{pmatrix} \in \underline{40}^* . \qquad (3.29)$$

Due to (2.4), the ideal $\hat{\pi}(\mathcal{J}^2\mathfrak{a})$ is given as the set of elements j_2 of the form

$$j_2 = \sum\nolimits_{\alpha,z\geq0}[\hat{\pi}(a^z_\alpha),[\dots[\hat{\pi}(a^1_\alpha),[\mathcal{M}^2,\hat{\pi}(a^0_\alpha)]]\dots]] \ , \text{ where} \qquad (3.30a)$$

$$0 = \sum\nolimits_{\alpha,z\geq0}[\hat{\pi}(a^z_\alpha),[\dots[\hat{\pi}(a^1_\alpha),[-\mathrm{i}\mathcal{M},\hat{\pi}(a^0_\alpha)]]\dots]] \ . \qquad (3.30b)$$

Obviously, terms in $\mathcal{M}^2$ proportional to the identities $\mathbb{1}_{10}, \mathbb{1}_5, 1$ do not contribute to j_2 . Next, the term $(\mathcal{M}^2)_{5,1} = \frac{3\mathrm{i}}{5}\pi_{5,1}(n) \otimes {M'_5}^TM'_e$ gives a contribution to j_2 , which is $\frac{3\mathrm{i}}{5} \otimes {M'_5}^T$ times (from the left) the contribution of $-\mathrm{i}\mathcal{M}_{5,1} = \pi_{5,1}(n) \otimes M'_e$ to (3.30b), and hence equals zero. For the same argument, all terms in $(\mathcal{M}^2)_{10,10}$ and $(\mathcal{M}^2)_{10,5}$ do not contribute to j_2 . The same is true for the terms proportional to $\pi_{10}(m)$ and $\pi_5(m)$. Thus, there remain only contributions from the terms

$-\mathrm{i}\pi_{10}(\frac{m}{2}+I_3)\otimes M^2_{A,10}\,,\ -\mathrm{i}\pi_5(\frac{m}{2}+I_3)\otimes M^2_{A,5}\,,\ -\mathrm{i}v_0\otimes M^2_V$ and $-\mathrm{i}n''\otimes M'_N\overline{M'_n}\,,$ where

$$M^2_V := {M'_{10}}^2 - 2(M'_{\tilde u}{M'_{\tilde n}}^* + M'_{\tilde n}{M'_{\tilde u}}^* - 2M'_{\tilde n}{M'_{\tilde n}}^*)\;, \tag{3.31a}$$
$$M^2_{A,10} := M'_u{M'_u}^* - M'_d{M'_d}^*\;, \qquad M^2_{A,5} := \bar M'_e{M'_e}^T - {M'_n}^*M'_n\;. \tag{3.31b}$$

Since the irreducible representations $\underline{24},\underline{75},\underline{5},\underline{45}^*,\underline{50},\underline{40}^*$ are independent, it is always possible to fulfil (3.30b) and to generate by the commutators (3.30a) representations of arbitrary elements of $\underline{75}$ and $\underline{40}^*$. Moreover, it can be checked that the generator $\frac{m}{2}+I_3$ occurring in $\mathcal{M}^2$ generates independent elements of the $\underline{24}$–representation. Hence, $j_2\in J_2:=\hat\pi(\mathcal{J}^2\mathfrak{a})$ takes the form

$$j_2 = \left(\begin{array}{ccc|ccc} \begin{bmatrix}\mathrm{i}\pi_{10}(a)\otimes M^2_{A,10}\\ +\mathrm{i}v\otimes M^2_V\end{bmatrix} & \mathrm{i}c''\otimes M'_N\overline{M'_n} & 0 & & & \\ (\mathrm{i}c''\otimes M'_N\overline{M'_n})^* & (\mathrm{i}\pi_5(a)\otimes M^2_{A,5})^T & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \\ \hline & & & \begin{bmatrix}(\mathrm{i}\pi_{10}(a)\otimes M^2_{A,10}\\ +\mathrm{i}v\otimes M^2_V)^T\end{bmatrix} & \overline{\mathrm{i}c''\otimes M'_N\overline{M'_n}} & 0 \\ & \mathrm{O} & & (\mathrm{i}c''\otimes M'_N\overline{M'_n})^T & \mathrm{i}\pi_5(a)\otimes M^2_{A,5} & 0 \\ & & & 0 & 0 & 0 \end{array}\right), \tag{3.32}$$

where $a\in\mathfrak{a}\,,\ v\in\mathfrak{v}$ and $c''\in\underline{40}^*$.

Let $J_0:=\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}$. From (3.3) and (3.24) we conclude that elements $j_0\in J_0$ are of the form

$$j_0 = \left(\begin{array}{ccc|ccc} \frac{3}{5}\alpha\mathbb{1}_{10}+\frac{1}{3}\mathrm{i}\pi_{10}(a)+\mathrm{i}v & 0 & 0 & & & \\ 0 & \frac{2}{5}\alpha\mathbb{1}_5+\mathrm{i}\pi_5(a)^T & 0 & & \mathrm{O} & \\ 0 & 0 & 0 & & & \\ \hline & & & \frac{3}{5}\alpha\mathbb{1}_{10}+\frac{1}{3}\mathrm{i}\pi_{10}(a)^T+\mathrm{i}v^T & 0 & 0 \\ & \mathrm{O} & & 0 & \frac{2}{5}\alpha\mathbb{1}_5+\mathrm{i}\pi_5(a) & 0 \\ & & & 0 & 0 & 0 \end{array}\right)\otimes\mathbb{1}_6\,, \tag{3.33}$$

where $\alpha\in\mathbb{R}\,,\ a\in\mathfrak{a}$ and $v\in\mathfrak{v}$.

It remains to find the spaces $\mathbb{j}^0\mathfrak{a},\mathbb{j}^1\mathfrak{a},\mathbb{j}^2\mathfrak{a}$ occurring in (2.6). For generic mass matrices $M_{\tilde u,d,e,\tilde\nu,10,5}$ the result is [12]

$$\mathbb{j}^0\mathfrak{a}=\{0\}\;,\qquad \mathbb{j}^1\mathfrak{a}=\{0\}\;,\qquad \mathbb{j}^2\mathfrak{a}=\hat\pi(\mathcal{J}^2\mathfrak{a})+\{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}+\mathbb{R}\mathbb{1}_{192}\;. \tag{3.34}$$

Let $J_3:=\mathbb{R}\mathbb{1}_{192}$. It is advantageous to construct an orthogonal decomposition $J_0+J_2+J_3=(J_0+J_3)\oplus J_2$. The result is that elements $j'_2\in J'_2$ are of the form

$$j_2' = \tag{3.35}$$
$$\left(\begin{array}{ccc|ccc}
\begin{bmatrix}\mathrm{i}\pi_{10}(a)\otimes M^2_{ud}\\ +\mathrm{i}v\otimes \tilde{M}^2_V\end{bmatrix} & \mathrm{i}c''\otimes M'_N\overline{M'_n} & 0 & & & \\
(\mathrm{i}c''\otimes M'_N\overline{M'_n})^* & (\mathrm{i}\pi_5(a)\otimes M^2_{en})^T & 0 & & \mathrm{O} & \\
0 & 0 & 0 & & & \\ \hline
& & & \begin{bmatrix}(\mathrm{i}\pi_{10}(a)\otimes M^2_{ud}\\ +\mathrm{i}v\otimes \tilde{M}^2_V)^T\end{bmatrix} & \overline{\mathrm{i}c''\otimes M'_N\overline{M'_n}} & 0 \\
& \mathrm{O} & & (\mathrm{i}c''\otimes M'_N\overline{M'_n})^T & \mathrm{i}\pi_5(a)\otimes M^2_{en} & 0 \\
& & & 0 & 0 & 0
\end{array}\right),$$

where

$$\begin{aligned}
M^2_{ud} &:= (M'_uM'^*_u - M'_dM'^*_d) - \tfrac{1}{24}\operatorname{tr}(M'_uM'^*_u - M'_dM'^*_d + \bar{M}'_eM'^T_e - M'^*_nM'_n)\mathbb{1}_6\,,\\
M^2_{en} &:= (\bar{M}'_eM'^T_e - M'^*_nM'_n) - \tfrac{1}{8}\operatorname{tr}(M'_uM'^*_u - M'_dM'^*_d + \bar{M}'_eM'^T_e - M'^*_nM'_n)\mathbb{1}_6\,,
\end{aligned} \tag{3.36a}$$
$$\tilde{M}^2_V := M^2_V - \tfrac{1}{6}\operatorname{tr}(M^2_V)\mathbb{1}_6\ . \tag{3.36b}$$

3.6. **The Factorization.** Due to (3.34), the problem of solving (2.9) is equivalent to finding for each given $\tau^2 \in \hat{\pi}(\Omega^2\mathfrak{a})$ an element $j \in J$ such that

$$\operatorname{tr}(\tilde{j}^*(\tau^2 + j)) = 0\ , \quad \forall \tilde{j} \in J\ . \tag{3.37}$$

Since J is block–diagonal, the off–diagonal blocks $\tau_{i,j}$ do not contribute to the trace (3.37). Next, in the parts $\pi_{10;5}(\mathrm{i}(b,b)')$ we can (and must) modulo J_2 replace

$$\begin{aligned}
M'_{\tilde{u}}M'^*_{\tilde{u}} - M'_dM'^*_d &\mapsto M'_{\tilde{u}}M'^*_{\tilde{u}} - M'_uM'^*_u = -M'_{\tilde{u}}M'^*_{\tilde{n}} - M'_{\tilde{n}}M'^*_{\tilde{u}} - M'_{\tilde{n}}M'^*_{\tilde{n}}\ ,\\
\bar{M}'_eM'^T_e - M'^*_{\tilde{u}}M'_{\tilde{u}} &\mapsto M'^*_nM'_n - M'^*_{\tilde{u}}M'_{\tilde{u}} = -3M'^*_{\tilde{u}}M'_{\tilde{n}} - 3M'^*_{\tilde{n}}M'_{\tilde{u}} + 9M'^*_{\tilde{n}}M'_{\tilde{n}}\ ,
\end{aligned} \tag{3.38}$$

see (3.27). In the diagonal part (3.21b) of τ^2 let us define

$$\begin{aligned}
A^{10} &:= \tfrac{1}{2}\{\pi_{10}(a),\pi_{10}(a)\}\ , & A^5 &:= \tfrac{1}{2}\{\pi_5(a),\pi_5(a)\}\ ,\\
B &:= -b^*b\ , & (b,b)' &:= bb^* - \tfrac{1}{5}\operatorname{tr}(bb^*)\mathbb{1}_5\ ,\\
U^{10} &:= -cc^*\ , & \tilde{U}^{10} &:= -c\pi_{10,10}(b)^*\\
\tilde{V}^{10} &:= -ww^*\ , & V^{10} &:= \tilde{V}^{10} - \mathrm{i}\pi_{10}(\mathrm{i}(b,b)')\ ,\\
\tilde{V}^5 &:= -w^*w\ , & V^5 &:= \tilde{V}^5 + 9\mathrm{i}\pi_5(\mathrm{i}(b,b)')\ ,\\
\tilde{W}^{10} &:= -\pi_{10,5}(b)w^*\ , & W^{10} &:= \tilde{W}^{10} - \mathrm{i}\pi_{10}(\mathrm{i}(b,b)')\ ,\\
\tilde{W}^5 &:= -w^*\pi_{10,5}(b)\ , & W^5 &:= \tilde{W}^5 - 3\mathrm{i}\pi_5(\mathrm{i}(b,b)')\ .
\end{aligned} \tag{3.39}$$

It is necessary to split $A^{10}, U^{10}, \tilde{U}^{10}, V^{10}$ and W^{10} according to (3.2a) and A^5, V^5 and W^5 according to (3.2b) into irreducible components. It turns out that the non–vanishing components are

$$\begin{aligned}
A^{10} &= A^{10}_{\underline{1}} \oplus A^{10}_{\underline{24}} \oplus A^{10}_{\underline{75}}\ , & A^5 &= A^5_{\underline{1}} \oplus A^5_{\underline{24}}\ ,\\
U^{10} &= U^{10}_{\underline{1}} \oplus U^{10}_{\underline{24}} \oplus U^{10}_{\underline{75}}\ , & \tilde{U}^{10} &= \tilde{U}^{10}_{\underline{75}}\ ,\\
V^{10} &= V^{10}_{\underline{1}} \oplus V^{10}_{\underline{24}} \oplus V^{10}_{\underline{75}}\ , & V^5 &= V^5_{\underline{1}} \oplus V^5_{\underline{24}}\ ,\\
W^{10} &= W^{10}_{\underline{24}} \oplus W^{10}_{\underline{75}}\ , & W^5 &= W^5_{\underline{24}}\ .
\end{aligned} \tag{3.40}$$

For these components we find

$$\begin{array}{lll} A^{10}_{\underline{1}} = \frac{3}{10}\ \mathrm{tr}(A^5)\mathbb{1}_{10}\ , & A^5_{\underline{1}} = \frac{1}{5}\ \mathrm{tr}(A^5)\mathbb{1}_5\ , & A^{10}_{\underline{75}} = A^{10} - A^{10}_{\underline{24}} - A^{10}_{\underline{1}}\ , \\ A^5_{\underline{24}} = A^5 - \frac{1}{5}\,\mathrm{tr}(A^5)\mathbb{1}_5\ , & & A^{10}_{\underline{24}} = -\frac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}A^5_{\underline{24}})\ , \\ U^{10}_{\underline{1}} = \frac{1}{10}\ \mathrm{tr}(U^{10})\mathbb{1}_{10}\ , & W^{10}_{\underline{75}} = W^{10} - W^{10}_{\underline{24}}\ , & U^{10}_{\underline{75}} \equiv U_{10} - U^{10}_{\underline{1}} - U^{10}_{\underline{24}}\ , \\ V^{10}_{\underline{1}} = \frac{1}{10}\ \mathrm{tr}(\tilde{V}^5)\mathbb{1}_{10}\ , & V^5_{\underline{1}} = \frac{1}{5}\ \mathrm{tr}(\tilde{V}^5)\mathbb{1}_5\ , & V^{10}_{\underline{75}} = V^{10} - V^{10}_{\underline{24}} - V^{10}_{\underline{1}}\ , \\ V^5_{\underline{24}} = \tilde{V}^5 - \frac{1}{5}\,\mathrm{tr}(\tilde{V}^5)\mathbb{1}_5 + 9\mathrm{i}\pi_5(\mathrm{i}(b,b)')\ , & & V^{10}_{\underline{24}} = -\mathrm{i}\pi_{10}(\mathrm{i}\tilde{V}'_{\underline{24}} + \mathrm{i}(b,b)')\ , \\ W^{10}_{\underline{24}} + W^{10*}_{\underline{24}} = -\frac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}W^5 + \mathrm{i}W^{5*})\ , & & W^{10}_{\underline{24}} - W^{10*}_{\underline{24}} = \frac{1}{3}\pi_{10}(W^5 - W^{5*})\ . \end{array} \tag{3.41}$$

Here, the term $\tilde{V}'_{\underline{24}} \in \mathrm{i}\mathfrak{a}$ is obtained as follows: We decompose $\mathrm{i}\tilde{V}'_{\underline{24}} := \sum_{j=1}^{24} a_j\beta_j$, where $\{\beta_j\}_{j=1,\dots,24}$ is an orthonormal basis of $\mathfrak{a}$, $\mathrm{tr}(\beta_i\beta_j) = -\delta_{ij}$. One finds the equation $a_j = -\frac{1}{3}\,\mathrm{tr}(\pi_{10}(\mathrm{i}\tilde{V}'_{\underline{24}})\pi_{10}(\beta_j)) \equiv -\frac{1}{3}\,\mathrm{tr}(\mathrm{i}\tilde{V}^{10}\pi_{10}(\beta_j))$, therefore,

$$\mathrm{i}\tilde{V}'_{\underline{24}} = -\tfrac{1}{3}\textstyle\sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}\tilde{V}^{10}\pi_{10}(\beta_j))\beta_j\ . \tag{3.42}$$

This formula shows the way how to obtain the other formulae of (3.41). One can prove

$$\begin{array}{c} \sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}A^{10}\pi_{10}(\beta_j)) \equiv \sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}A^5\pi_5(\beta_j))\ , \\ \sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}\tilde{W}^{10}\pi_{10}(\beta_j)) \equiv \sum_{j=1}^{24}\mathrm{tr}(\mathrm{i}\tilde{W}^5\pi_5(\beta_j))\ . \end{array} \tag{3.43}$$

However, there is no such simple relation between the 10– and 5–components of the $\tilde{V}$–part.

Due to (3.31a) we can modulo J_2 replace $A^{10}_{\underline{75}} \otimes {M'_{10}}^2$ by

$$A^{10}_{\underline{75}} \otimes (2M'_{\tilde{n}}{M'_{\tilde{u}}}^* + 2M'_{\tilde{u}}{M'_{\tilde{n}}}^* - 4M'_{\tilde{n}}{M'_{\tilde{n}}}^* - \tfrac{1}{2}M'_N{M'_N}^*)\ . \tag{3.44}$$

Now we add to τ^2 the element $j_0 \in J_0$ given by

$$\begin{aligned} \alpha &= \mathrm{tr}(A^5)\alpha_A + B\alpha_B + \mathrm{tr}(U^{10})\alpha_U + \mathrm{tr}(V^5)\alpha_V\ , \\ \mathrm{i}a &= A^5_{\underline{24}}\beta_A - \mathrm{i}\pi_{10}^{-1}(\mathrm{i}U^{10}_{\underline{24}})\beta_U + V^5_{\underline{24}}\check{\beta}_V - \mathrm{i}\pi_{10}^{-1}(\mathrm{i}V^{10}_{\underline{24}})\beta_V \\ &\quad + (W^5 + W^{5*})\beta_W + \mathrm{i}(W^5 - W^{5*})\beta'_W\ , \\ \mathrm{i}v &= (V^{10}_{\underline{75}} - 4A^{10}_{\underline{75}})\gamma_V + (U^{10}_{\underline{75}} - \tfrac{1}{2}A^{10}_{\underline{75}})\gamma_U + (\tilde{U}^{10}_{\underline{75}} + \tilde{U}^{10*}_{\underline{75}})\tilde{\gamma}_U + \mathrm{i}(\tilde{U}^{10}_{\underline{75}} - \tilde{U}^{10*}_{\underline{75}})\tilde{\gamma}'_U \\ &\quad + (W^{10}_{\underline{75}} + W^{10*}_{\underline{75}} + 4A^{10}_{\underline{75}})\gamma_W + \mathrm{i}(W^{10}_{\underline{75}} - W^{10*}_{\underline{75}})\gamma'_W\ . \end{aligned} \tag{3.45a}$$

Moreover, we add the element $j_2 \in J'_2$ given by

$$\begin{aligned} \mathrm{i}a &= A^5_{\underline{24}}\delta_A - \mathrm{i}\pi_{10}^{-1}(\mathrm{i}U^{10}_{\underline{24}})\delta_U + V^5_{\underline{24}}\check{\epsilon}_V - \mathrm{i}\pi_{10}^{-1}(\mathrm{i}V^{10}_{\underline{24}})\delta_V \\ &\quad + (W^5 + W^{5*})\delta_W + \mathrm{i}(W^5 - W^{5*})\delta'_W\ , \\ \mathrm{i}v &= (V^{10}_{\underline{75}} - 4A^{10}_{\underline{75}})\epsilon_V + (U^{10}_{\underline{75}} - \tfrac{1}{2}A^{10}_{\underline{75}})\epsilon_U + (U^{10}_{\underline{75}} + U^{10*}_{\underline{75}})\tilde{\epsilon}_U + \mathrm{i}(U^{10}_{\underline{75}} - U^{10*}_{\underline{75}})\tilde{\epsilon}'_U \\ &\quad + (W^{10}_{\underline{75}} + W^{10*}_{\underline{75}} + 4A^{10}_{\underline{75}})\epsilon_W + \mathrm{i}(W^{10}_{\underline{75}} - W^{10*}_{\underline{75}})\epsilon'_W\ , \end{aligned} \tag{3.45b}$$

and the element $j_3 \in J_3$ determined by

$$\nu = \mathrm{tr}(A^5)\zeta_A + B\zeta_B + \mathrm{tr}(U^{10})\zeta_U + \mathrm{tr}(V^5)\zeta_V\ . \tag{3.45c}$$

As result, the matrix elements $\hat\tau_{10},\hat\tau_5,\hat\tau_1$ of $\hat\tau^2=\tau^2+j_0+j_2'+j_3$ take the form

$$\begin{aligned}
\hat\tau_{10} &= \mathrm{tr}(A^5)\mathbb{1}_{10}\otimes\hat M^{10}_{aa}+\mathrm{tr}(U^{10})\mathbb{1}_{10}\otimes\hat M^{10}_{cc}+\mathrm{tr}(V^5)\mathbb{1}_{10}\otimes\hat M^{10}_{nn}+B\mathbb{1}_{10}\otimes\hat M^{10}_{bb}\\
&\quad -\tfrac13\mathrm{i}\pi_{10}(\mathrm{i}A^5_{\underline{24}})\otimes M^{10}_{aa}+U^{10}_{\underline{24}}\otimes M^{10}_{cc}+V^{10}_{\underline{24}}\otimes M^{10}_{nn}-\mathrm{i}\pi_{10}(\mathrm{i}V^5_{\underline{24}})\otimes\check M^{10}_{nn}\\
&\qquad -\tfrac13\mathrm{i}\pi_{10}(\mathrm{i}W^5+\mathrm{i}W^{5*})\otimes M^{10}_{\{un\}}+\tfrac13\mathrm{i}\pi_{10}(W^5-W^{5*})\otimes M^{10}_{[un]}\\
&\quad +(V^{10}_{\underline{75}}-4A^{10}_{\underline{75}})\otimes\tilde M^{10}_{nn}+(U^{10}_{\underline{75}}-\tfrac12A^{10}_{\underline{75}})\otimes\tilde M^{10}_{cc}+(\tilde U^{10}_{\underline{75}}+\tilde U^{10}_{\underline{75}})\otimes\tilde M^{10}_{\{cd\}}\\
&\qquad +\mathrm{i}(\tilde U^{10}_{\underline{75}}-\tilde U^{10}_{\underline{75}})\otimes\tilde M^{10}_{[cd]}+(W^{10}_{\underline{75}}+W^{10*}_{\underline{75}}+4A^{10}_{\underline{75}})\otimes\tilde M^{10}_{\{un\}}\\
&\qquad +\mathrm{i}(W^{10}_{\underline{75}}-W^{10*}_{\underline{75}})\otimes\tilde M^{10}_{[un]} \qquad\qquad (3.46)\\
\hat\tau_5 &= \mathrm{tr}(A^5)\mathbb{1}_5\otimes\hat M^5_{aa}+\mathrm{tr}(U^{10})\mathbb{1}_5\otimes\hat M^5_{cc}+\mathrm{tr}(V^5)\mathbb{1}_5\otimes\hat M^5_{nn}+B\mathbb{1}_5\otimes\hat M^5_{bb}\\
&\quad +A^5_{\underline{24}}\otimes M^5_{aa}+V^5_{\underline{24}}\otimes\check M^5_{nn}-\mathrm{i}\pi^{-1}_{10}(\mathrm{i}V^{10}_{\underline{24}})\otimes M^5_{nn}-\mathrm{i}\pi^{-1}_{10}(\mathrm{i}U^{10}_{\underline{24}})\otimes M^5_{cc}\\
&\qquad +(W^5+W^{5*})\otimes M^5_{\{un\}}+\mathrm{i}(W^5-W^{5*})\otimes M^5_{[un]}\\
\hat\tau_1 &= \mathrm{tr}(A^5)\otimes\hat M^1_{aa}+\mathrm{tr}(U^{10})\otimes\hat M^1_{cc}+B\otimes\hat M^1_{bb}+\mathrm{tr}(V^5)\otimes\hat M^1_{nn}\;,
\end{aligned}$$

where the matrices $M^k_{ij},\tilde M^k_{ij},\hat M^k_{ij}$ and $\check M^k_{ij}$ are given in Appendix A.

The last step before including the function algebra is to apply the map $\hat\sigma\circ\hat\pi^{-1}$ defined in (2.2b) to elements $\tau^1\in\hat\pi(\Omega^1\mathfrak{a})$. Calculating $\hat\sigma\circ\hat\pi^{-1}(\tau^1)$ means to calculate j_2 in (3.30a), however with the r.h.s. of (3.30b) equal to the given element τ^1 and not equal to zero. We have listed the matrix elements of $\mathcal{M}^2$ in (3.28). Again, terms in $\mathcal{M}^2$ proportional to the identities $\mathbb{1}_{10},\mathbb{1}_5,1$ do not contribute to j_2. Next, the terms proportional to $-\mathrm{i}v_0$, $-\mathrm{i}\pi_{10;5}(\frac m2+I_3)$ and $-\mathrm{i}n''$ contribute to the ideal $\hat\pi(\mathcal{J}^2\mathfrak{a})$, as explained above. Since we regard $\hat\sigma\circ\hat\pi^{-1}(\tau^1)$ modulo $\hat\pi(\mathcal{J}^2\mathfrak{a})$, it is not necessary to consider these terms. Therefore, there remain only the terms

$$\begin{aligned}
&-\mathrm{i}\pi_{10}(m)\otimes\tfrac13(\tfrac15M_{10}'^{\,2}-4M'_{\tilde u}M_{\tilde n}'^{\,*}-4M'_{\tilde n}M_{\tilde u}'^{\,*}+8M'_{\tilde n}M_{\tilde n}'^{\,*}+M'_NM_N'^{\,*})\;, &(3.47a)\\
&-\mathrm{i}\pi_5(m)\otimes(\tfrac15M_5'^2-4M_{\tilde n}'^{*}M'_{\tilde u}-4M_{\tilde u}'^{*}M'_{\tilde n}+8M_{\tilde n}'^{*}M'_{\tilde n}) &(3.47b)
\end{aligned}$$

in the diagonal blocks $(\mathcal{M}^2)_{10}$ and $(\mathcal{M}^2)_5$ as well as the off–diagonal blocks $(\mathcal{M}^2)_{i,j}$, which give a contribution to $\hat\sigma\circ\hat\pi^{-1}(\tau^1)$. As we have already noticed, the contribution of $(\mathcal{M}^2)_{5,1}$ is $\frac{3\mathrm{i}}{5}\otimes M_5'^T$ times the contribution of $\pi_{5,1}(n)\otimes M'_e$ to (3.30b). We get analogous contributions from the other terms $(\mathcal{M}^2)_{i,j}$ and $(\mathcal{M}^2)_i$. Thus, we obtain in the same notations as in (3.19a) the formula

$$\hat\sigma\circ\hat\pi^{-1}(\tau^1)=\left(\begin{array}{ccc|ccc}
\sigma_{10} & 0 & 0 & \sigma_{10,10} & \sigma_{10,5} & 0\\
0 & \sigma_5^T & 0 & \sigma_{10,5}^T & 0 & \sigma_{5,1}\\
0 & 0 & 0 & 0 & \sigma_{5,1}^T & 0\\
\hline
\sigma^*_{10,10} & \overline{\sigma_{10,5}} & 0 & \sigma_{10}^T & 0 & 0\\
\sigma^*_{10,5} & 0 & \overline{\sigma_{5,1}} & 0 & \sigma_5 & 0\\
0 & \sigma^*_{5,1} & 0 & 0 & 0 & 0
\end{array}\right)\;,\text{ where}\qquad(3.48)$$

$$\begin{aligned}
\sigma_{10} &= -\mathrm{i}\pi_{10}(a)\otimes\tfrac13(\tfrac15M_{10}'^{\,2}-4M'_{\tilde u}M_{\tilde n}'^{\,*}-4M'_{\tilde n}M_{\tilde u}'^{\,*}+8M'_{\tilde n}M_{\tilde n}'^{\,*}+M'_NM_N'^{\,*})\;,\\
\sigma_5 &= -\mathrm{i}\pi_5(a)\otimes(\tfrac15M_5'^2-4M_{\tilde n}'^{*}M'_{\tilde u}-4M_{\tilde u}'^{*}M'_{\tilde n}+8M_{\tilde n}'^{*}M'_{\tilde n})\;,\\
\sigma_{10,10} &= \tfrac{3\mathrm{i}}{5}\pi_{10,10}(b)\otimes\tfrac12(M'_{10}M'_d+M'_dM_{10}'^{\,T})\\
&\quad +\tfrac{\mathrm{i}}{2}\pi_{10,10}(w)\otimes\tfrac12(M'_{10}M'_d-M'_dM_{10}'^{\,T})-\tfrac{12\mathrm{i}}{5}c\otimes\tfrac12(M'_{10}M'_N+M'_N\overline{M'_{10}})\;,
\end{aligned}$$

$$
\begin{aligned}
\sigma_{5,1} &= \tfrac{3\mathrm{i}}{5}\pi_{5,1}(b)\otimes M_5'^{T}M_e' \;,\\
\sigma_{10,5} &= \mathrm{i}\pi_{10,5}(b)\otimes\big(-\tfrac{9}{20}M_{10}'M_{\tilde u}' - \tfrac{3}{20}M_{\tilde u}'M_5' + \tfrac{3}{4}M_{10}'M_{\tilde n}' - \tfrac{3}{4}M_{\tilde n}'M_5'\big)\\
&\quad + \mathrm{i}w\otimes\big(\tfrac{1}{4}M_{10}'M_{\tilde u}' - \tfrac{1}{4}M_{\tilde u}'M_5' - \tfrac{19}{20}M_{10}'M_{\tilde n}' + \tfrac{7}{20}M_{\tilde n}'M_5'\big)\;.
\end{aligned}
$$

Now, it remains to perform the factorization in the diagonal blocks (3.47a) and (3.47b). The same method as before yields that the representatives orthogonal to $J_2'\oplus(J_0+J_3)$ are

$$
(3.47\mathrm{a})\mapsto -\tfrac{1}{3}\mathrm{i}\pi_{10}(a)\otimes\big(\tfrac{1}{5}M_{aa}^{10} - 8M_{\{un\}}^{10} + 8M_{nn}^{10} + 24\check M_{nn}^{10} + M_{cc}^{10}\big)\;, \tag{3.49a}
$$
$$
(3.47\mathrm{b})\mapsto -\mathrm{i}\pi_{5}(a)\otimes\big(\tfrac{1}{5}M_{aa}^{5} - 8M_{\{un\}}^{5} + \tfrac{8}{3}M_{nn}^{5} + 8\check M_{nn}^{5} + \tfrac{1}{3}M_{cc}^{5}\big)\;. \tag{3.49b}
$$

4. The Action of the Unification Model

4.1. **The Curvature.** Now we are able to construct the bosonic action of the flipped SU(5) × U(1)–grand unification model. We choose X to be a four dimensional Riemannian spin manifold. When using a local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1 then the basis elements γ^μ are selfadjoint as complex sections of the Clifford bundle. Elements of Λ^1 are locally represented by real linear combinations of $\{\gamma^\mu\}_{\mu=1,2,3,4}$. The grading operator is $\gamma^5=\gamma^1\gamma^2\gamma^3\gamma^4$.

The first step is to write down the connection form ρ, which has according to (3.25) the structure

$$
\begin{gathered}
\rho=\pi(A)+\pi(A'')+\gamma^5\pi(H)\;,\\
A\in\Lambda^1\otimes\mathrm{su}(5)\;,\quad A''\in\Lambda^1\otimes\mathrm{u}(1)\;,\quad H\in\Lambda^0\otimes\Omega^1\mathfrak{a}\;.
\end{gathered} \tag{4.1}
$$

Here, γ^5 acts componentwise and $\pi=\mathrm{id}\otimes\hat\pi$, where the matrix parts of $\pi(A)$ and $\pi(A'')$ are given by (3.3) and (3.23b), respectively. Elements of $\hat\pi(\Omega^1\mathfrak{a})$ are specified by elements of $\mathfrak{a},\mathfrak{b},\mathfrak{c}$ and $\mathfrak{w}$, see (3.19). Thus, we consider H as a sum

$$
\begin{gathered}
H=\Psi+\Phi+\Xi+\Upsilon\;,\\
\Psi\in\Lambda^0\otimes\mathfrak{a}\;,\quad \Phi\in\Lambda^0\otimes\mathfrak{b}\;,\quad \Xi\in\Lambda^0\otimes\mathfrak{c}\;,\quad \Upsilon\in\Lambda^0\otimes\mathfrak{w}\;.
\end{gathered} \tag{4.2}
$$

Inserting (4.1) and (4.2) into formula (2.8) for the curvature, we find with (3.48)

$$
\begin{aligned}
\theta &= \mathbf{d}\pi(A)+\mathbf{d}\pi(A'')+\tfrac{1}{2}\{\pi(A),\pi(A)\}\\
&\quad -\gamma^5\big(\mathbf{d}\pi(\Psi)+\mathbf{d}\pi(\Phi)+\mathbf{d}\pi(\Xi)+\mathbf{d}\pi(\Upsilon)\\
&\qquad +[\pi(A)+\pi(A''),\pi(\Psi)+\pi(\Phi)+\pi(\Xi)+\pi(\Upsilon)-\mathrm{i}\mathcal{M}]\big)\\
&\quad +\big(\tfrac{1}{2}\{\pi(\Psi)+\pi(\Phi)+\pi(\Xi)+\pi(\Upsilon),\pi(\Psi)+\pi(\Phi)+\pi(\Xi)+\pi(\Upsilon)\}\\
&\qquad +\{\pi(\Psi)+\pi(\Phi)+\pi(\Xi)+\pi(\Upsilon),-\mathrm{i}\mathcal{M}\}+\hat\sigma_{\mathfrak{g}}(\rho)\quad \mathrm{mod}\ \Lambda^0\otimes\mathbb{j}^2\mathfrak{a}\big)\;,
\end{aligned} \tag{4.3a}
$$

where

$$\begin{aligned}
\hat{\sigma}_{\mathfrak{g}}(\rho) := & -\tfrac{12\mathrm{i}}{5}\pi(\Xi\otimes\tfrac{1}{2}(M'_{10}M'_N+M'_NM'^{\,T}_{10})) + \tfrac{\mathrm{i}}{2}\pi(\pi_{10,10}(\Upsilon)\otimes\tfrac{1}{2}(M'_{10}M'_d-M'_dM'^{\,T}_{10}))\\
&+ \tfrac{3\mathrm{i}}{5}\pi(\pi_{10,10}(\Phi)\otimes\tfrac{1}{2}(M'_{10}M'_d+M'_dM'^{\,T}_{10}) + \tfrac{3\mathrm{i}}{5}\pi(\pi_{5,1}(\Phi)\otimes M'^{\,T}_5M'_e)\\
&- \mathrm{i}\pi(\pi_{10,5}(\Phi)\otimes(\tfrac{9}{20}M'_{10}M'_{\tilde{u}}+\tfrac{3}{20}M'_{\tilde{u}}M'_5-\tfrac{3}{4}M'_{10}M'_{\tilde{n}}+\tfrac{3}{4}M'_{\tilde{n}}M'_5))\\
&- \mathrm{i}\pi(\pi_{10,5}(\Upsilon)\otimes(-\tfrac{1}{4}M'_{10}M'_{\tilde{u}}+\tfrac{1}{4}M'_{\tilde{u}}M'_5+\tfrac{19}{20}M'_{10}M'_{\tilde{n}}-\tfrac{7}{20}M'_{\tilde{n}}M'_5))\\
&- \tfrac{1}{3}\mathrm{i}\pi(\pi_{10}(\Psi)\otimes(\tfrac{1}{5}M^{10}_{aa}-8M^{10}_{\{un\}}+8M^{10}_{nn}+24\check{M}^{10}_{nn}+M^{10}_{cc}))\\
&- \mathrm{i}\pi(\pi_5(\Psi)\otimes(\tfrac{1}{5}M^5_{aa}-8M^5_{\{un\}}+\tfrac{8}{3}M^5_{nn}+8\check{M}^5_{nn}+\tfrac{1}{3}M^5_{cc}))\;.
\end{aligned}\tag{4.3b}$$

Here we have denoted by π the embedding of the selected matrix elements of (3.48) into the matrix (3.48). We have

$$\begin{aligned}
&\tfrac{1}{2}\{\,\pi(\Psi)+\pi(\Phi)+\pi(\Xi)+\pi(\Upsilon),\pi(\Psi)+\pi(\Phi)+\pi(\Xi)+\pi(\Upsilon)\}\\
&\quad+\{\pi(\Psi)+\pi(\Phi)+\pi(\Xi)+\pi(\Upsilon),-\mathrm{i}\mathcal{M}\}\\
&\quad=\tfrac{1}{2}\{\pi(\tilde{\Psi})+\pi(\tilde{\Phi})+\pi(\tilde{\Xi})+\pi(\tilde{\Upsilon}),\pi(\tilde{\Psi})+\pi(\tilde{\Phi})+\pi(\tilde{\Xi})+\pi(\tilde{\Upsilon})\}+\mathcal{M}^2\;,
\end{aligned}\tag{4.4a}$$

where

$$\tilde{\Psi}:=\Psi+m\;,\qquad \tilde{\Phi}:=\Phi+n\;,\qquad \tilde{\Xi}:=\Xi+m'\;,\qquad \tilde{\Upsilon}:=\Upsilon+n'\;.\tag{4.4b}$$

Let

$$\hat{\sigma}_{\mathfrak{g}}(\tilde{\rho}):=\text{ formula (4.3b) with }\Psi\mapsto\tilde{\Psi}\;,\;\Phi\mapsto\tilde{\Phi}\;,\;\Xi\mapsto\tilde{\Xi}\;,\;\Upsilon\mapsto\tilde{\Upsilon}\;.\tag{4.5}$$

Then we obtain from (4.3a) and (3.28)

$$\begin{aligned}
\theta = {} & \mathbf{d}\pi(A)+\tfrac{1}{2}\{\pi(A),\pi(A)\}\\
&-\gamma^5(\mathbf{d}\pi(\tilde{\Psi})+\mathbf{d}\pi(\tilde{\Phi})+\mathbf{d}\pi(\tilde{\Upsilon})+\mathbf{d}\pi(\tilde{\Xi})\\
&+[\pi(A)+\pi(A''),\pi(\tilde{\Psi})+\pi(\tilde{\Phi})+\pi(\tilde{\Xi})+\pi(\tilde{\Upsilon})])\\
&+\big(\tfrac{1}{2}\{\pi(\tilde{\Psi})+\pi(\tilde{\Phi})+\pi(\tilde{\Xi})+\pi(\tilde{\Upsilon}),\pi(\tilde{\Psi})+\pi(\tilde{\Phi})+\pi(\tilde{\Xi})+\pi(\tilde{\Upsilon})\}+\hat{\sigma}_{\mathfrak{g}}(\tilde{\rho})\\
&+\mathrm{diag}\,\big(\mathbb{1}_{10}\otimes(\tfrac{6}{5}\hat{M}^{10}_{aa}+\hat{M}^{10}_{bb}+12\hat{M}^{10}_{nn}+\hat{M}^{10}_{cc})\,,\;\mathbb{1}_5\otimes(\tfrac{6}{5}\hat{M}^5_{aa}+\hat{M}^5_{bb}+12\hat{M}^5_{nn}+\hat{M}^5_{cc})^T\,,\\
&\qquad\tfrac{6}{5}\hat{M}^1_{aa}+\hat{M}^1_{bb}+12\hat{M}^1_{nn}+\hat{M}^1_{cc}\,,\;\mathbb{1}_{10}\otimes(\tfrac{6}{5}\hat{M}^{10}_{aa}+\hat{M}^{10}_{bb}+12\hat{M}^{10}_{nn}+\hat{M}^{10}_{cc})^T\,,\\
&\qquad\mathbb{1}_5\otimes(\tfrac{6}{5}\hat{M}^5_{aa}+\hat{M}^5_{bb}+12\hat{M}^5_{nn}+\hat{M}^5_{cc})\,,\;(\tfrac{6}{5}\hat{M}^1_{aa}+\hat{M}^1_{bb}+12\hat{M}^1_{nn}+\hat{M}^1_{cc})^T\,\big)\\
&\qquad\qquad\mod\Lambda^0\otimes\mathbb{j}^2\mathfrak{a}\big)\;.
\end{aligned}\tag{4.6}$$

We define

$$\check{\tilde{\Phi}}:=\pi_{10,5}(\tilde{\Phi})\;,\qquad\qquad \hat{\tilde{\Phi}}:=\pi_{10,10}(\tilde{\Phi})\;,$$
$$(\tilde{\Phi},\tilde{\Phi})':=\tilde{\Phi}\tilde{\Phi}^*-\tfrac{1}{5}\,\mathrm{tr}(\tilde{\Phi}\tilde{\Phi}^*)\mathbb{1}_5\;,\qquad (\tilde{\Xi}\tilde{\Xi}^*)':=-\mathrm{i}\pi_{10}^{-1}(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)_{\underline{24}})\;.$$

Using (3.46) and (3.21) we obtain the following matrix representation of $\mathfrak{e}(\theta)$:

$$\mathfrak{e}(\theta)=\left(\begin{array}{ccc|ccc}
\theta_{10} & \theta_{\widetilde{10,5}} & \theta_{10,1} & \theta_{10,10} & \theta_{10,5} & 0\\
\theta^*_{\widetilde{10,5}} & \theta^T_5 & 0 & \theta^T_{10,5} & 0 & \theta_{5,1}\\
\theta^*_{10,1} & 0 & \theta_1 & 0 & \theta^T_{5,1} & 0\\
\hline
\theta^*_{10,10} & \overline{\theta_{10,5}} & 0 & \theta^T_{10} & \overline{\theta_{\widetilde{10,5}}} & \overline{\theta_{10,1}}\\
\theta^*_{10,5} & 0 & \overline{\theta_{5,1}} & \theta^T_{\widetilde{10,5}} & \theta_5 & 0\\
0 & \theta^*_{5,1} & 0 & \theta^T_{10,1} & 0 & \theta^T_1
\end{array}\right)\;,\text{ where}\tag{4.7a}$$

$$
\begin{aligned}
\theta_{10} &= \pi_{10}(\mathbf{d}A + \tfrac{1}{2}\{A,A\}) \otimes \mathbb{1}_6 - \tfrac{1}{2}\mathbf{d}A''\mathbb{1}_{10} \otimes \mathbb{1}_6 - \gamma^5\pi_{10}(\mathbf{d}\tilde{\Psi} + [A,\tilde{\Psi}]) \otimes M'_{10} \\
&\quad + (\tfrac{6}{5} + \operatorname{tr}(\tilde{\Psi}^2))\mathbb{1}_{10} \otimes \hat{M}^{10}_{aa} + (1 - \tilde{\Phi}^*\tilde{\Phi})\mathbb{1}_{10} \otimes \hat{M}^{10}_{bb} \\
&\quad + (1 - \operatorname{tr}(\tilde{\Xi}\tilde{\Xi}^*))\mathbb{1}_{10} \otimes \hat{M}^{10}_{cc} + (12 - \operatorname{tr}(\tilde{\Upsilon}\tilde{\Upsilon}^*))\mathbb{1}_{10} \otimes \hat{M}^{10}_{nn} \\
&\quad - \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Psi}^2 - \tfrac{1}{5}\operatorname{tr}(\tilde{\Psi}^2)\mathbb{1}_5 - \tfrac{1}{5}\mathrm{i}\tilde{\Psi})) \otimes M^{10}_{aa} + \mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)' - \tfrac{1}{3}\tilde{\Psi}) \otimes M^{10}_{cc} \\
&\quad + \mathrm{i}\pi_{10}(\mathrm{i}((\tilde{\Upsilon}\tilde{\Upsilon}^*)' + \tfrac{8}{3}\mathrm{i}\tilde{\Psi} - (\tilde{\Phi},\tilde{\Phi})')) \otimes M^{10}_{nn} \\
&\quad + \mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Upsilon}^*\tilde{\Upsilon} - \tfrac{1}{5}\operatorname{tr}(\tilde{\Upsilon}^*\tilde{\Upsilon})\mathbb{1}_5 + 8\mathrm{i}\tilde{\Psi} + 9(\tilde{\Phi},\tilde{\Phi})')) \otimes \check{M}^{10}_{nn} \\
&\quad + \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Upsilon}^*\check{\tilde{\Phi}} + \check{\tilde{\Phi}}^*\tilde{\Upsilon} - 8\mathrm{i}\tilde{\Psi} - 6(\tilde{\Phi},\tilde{\Phi})')) \otimes M^{10}_{\{un\}} - \tfrac{1}{3}\mathrm{i}\pi_{10}(\tilde{\Upsilon}^*\check{\tilde{\Phi}} - \check{\tilde{\Phi}}^*\tilde{\Upsilon}) \otimes M^{10}_{[un]} \\
&\quad - \big(\tilde{\Xi}\tilde{\Xi}^* + \tfrac{1}{2}(\pi_{10}(\tilde{\Psi}))^2 - \tfrac{1}{10}(\operatorname{tr}(\tilde{\Xi}\tilde{\Xi}^*) + \tfrac{3}{2}\operatorname{tr}(\tilde{\Psi}^2))\mathbb{1}_{10} \\
&\qquad + \mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)' + \tfrac{1}{6}\mathrm{i}(\tilde{\Psi}^2 - \tfrac{1}{5}\operatorname{tr}(\tilde{\Psi}^2)\mathbb{1}_5))\big) \otimes \tilde{M}^{10}_{cc} \\
&\quad - \big(\tilde{\Xi}\hat{\tilde{\Phi}}^* + \hat{\tilde{\Phi}}\tilde{\Xi}^*\big) \otimes \tilde{M}^{10}_{\{cd\}} - \mathrm{i}\big(\tilde{\Xi}\hat{\tilde{\Phi}}^* - \hat{\tilde{\Phi}}\tilde{\Xi}^*\big) \otimes \tilde{M}^{10}_{[cd]} \\
&\quad - \big(\tilde{\Upsilon}\tilde{\Upsilon}^* + 4(\pi_{10}(\tilde{\Psi}))^2 - \tfrac{1}{10}\operatorname{tr}(\tilde{\Upsilon}^*\tilde{\Upsilon} + 12\tilde{\Psi}^2)\mathbb{1}_{10} \\
&\qquad + \mathrm{i}\pi_{10}(\mathrm{i}((\tilde{\Upsilon}\tilde{\Upsilon}^*)' + \tfrac{4}{3}\tilde{\Psi}^2 - \tfrac{4}{15}\operatorname{tr}(\tilde{\Psi}^2)\mathbb{1}_5))\big) \otimes \tilde{M}^{10}_{nn} \\
&\quad - \big(\check{\tilde{\Phi}}\tilde{\Upsilon}^* + \tilde{\Upsilon}\check{\tilde{\Phi}}^* - 4(\pi_{10}(\tilde{\Psi}))^2 + \tfrac{6}{5}\operatorname{tr}(\tilde{\Psi}^2)\mathbb{1}_{10} \\
&\qquad + \tfrac{1}{3}\mathrm{i}\pi_{10}(\mathrm{i}(\tilde{\Upsilon}^*\check{\tilde{\Phi}} + \check{\tilde{\Phi}}^*\tilde{\Upsilon} - 4\tilde{\Psi}^2 + \tfrac{4}{5}\operatorname{tr}(\tilde{\Psi}^2)\mathbb{1}_5))\big) \otimes \tilde{M}^{10}_{\{un\}} \\
&\quad - \mathrm{i}\big(\check{\tilde{\Phi}}\tilde{\Upsilon}^* - \tilde{\Upsilon}\check{\tilde{\Phi}}^* - \tfrac{1}{3}\pi_{10}(\tilde{\Upsilon}^*\check{\tilde{\Phi}} - \check{\tilde{\Phi}}^*\tilde{\Upsilon})\big) \otimes \tilde{M}^{10}_{[un]} \;,
\end{aligned}
\tag{4.7b}
$$

$$
\begin{aligned}
\theta_5 &= \pi_5(\mathbf{d}A + \tfrac{1}{2}\{A,A\}) \otimes \mathbb{1}_6 - \tfrac{3}{2}\mathbf{d}A''\mathbb{1}_5 \otimes \mathbb{1}_6 + \gamma^5\pi_5(\mathbf{d}\tilde{\Psi} + [A,\tilde{\Psi}]) \otimes M'_5 \\
&\quad + (\tfrac{6}{5} + \operatorname{tr}(\tilde{\Psi}^2))\mathbb{1}_5 \otimes \hat{M}^5_{aa} + (1 - \tilde{\Phi}^*\tilde{\Phi})\mathbb{1}_5 \otimes \hat{M}^5_{bb} \\
&\quad + (1 - \operatorname{tr}(\tilde{\Xi}\tilde{\Xi}^*))\mathbb{1}_5 \otimes \hat{M}^5_{cc} + (12 - \operatorname{tr}(\tilde{\Upsilon}\tilde{\Upsilon}^*))\mathbb{1}_{10} \otimes \hat{M}^{10}_{nn} \\
&\quad - \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Psi}^2 - \tfrac{1}{5}\operatorname{tr}(\tilde{\Psi}^2)\mathbb{1}_5 - \tfrac{1}{5}\mathrm{i}\tilde{\Psi})) \otimes M^5_{aa} + \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Xi}\tilde{\Xi}^*)' - \tfrac{1}{3}\tilde{\Psi}) \otimes M^5_{cc} \\
&\quad + \mathrm{i}\pi_5(\mathrm{i}((\tilde{\Upsilon}\tilde{\Upsilon}^*)' + \tfrac{8}{3}\mathrm{i}\tilde{\Psi} - (\tilde{\Phi},\tilde{\Phi})')) \otimes M^5_{nn} \\
&\quad + \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Upsilon}^*\tilde{\Upsilon} - \tfrac{1}{5}\operatorname{tr}(\tilde{\Upsilon}^*\tilde{\Upsilon})\mathbb{1}_5 + 8\mathrm{i}\tilde{\Psi} + 9(\tilde{\Phi},\tilde{\Phi})')) \otimes \check{M}^5_{nn} \\
&\quad + \mathrm{i}\pi_5(\mathrm{i}(\tilde{\Upsilon}^*\check{\tilde{\Phi}} + \check{\tilde{\Phi}}^*\tilde{\Upsilon} - 8\mathrm{i}\tilde{\Psi} - 6(\tilde{\Phi},\tilde{\Phi})')) \otimes M^5_{\{un\}} - \mathrm{i}\pi_5(\tilde{\Upsilon}^*\check{\tilde{\Phi}} - \check{\tilde{\Phi}}^*\tilde{\Upsilon}) \otimes M^5_{[un]} \;,
\end{aligned}
\tag{4.7c}
$$

$$
\begin{aligned}
\theta_1 &= -\tfrac{5}{2}\mathbf{d}A''\mathbb{1}_6 + (\tfrac{6}{5} + \operatorname{tr}(\tilde{\Psi}^2)) \otimes \hat{M}^1_{aa} + (1 - \tilde{\Phi}^*\tilde{\Phi}) \otimes \hat{M}^1_{bb} \\
&\quad + (1 - \operatorname{tr}(\tilde{\Xi}\tilde{\Xi}^*)) \otimes \hat{M}^1_{cc} + \operatorname{tr}(12 - \operatorname{tr}(\Upsilon\Upsilon^*)) \otimes M^1_{nn} \;,
\end{aligned}
\tag{4.7d}
$$

$$
\begin{aligned}
\theta_{10,10} &= -\gamma^5\pi_{10,10}(\mathbf{d}\tilde{\Phi} + (A + A''\mathbb{1}_5)\tilde{\Phi}) \otimes M'_d \\
&\quad - \gamma^5(\mathbf{d}\tilde{\Xi} + \pi_{10}(A)\tilde{\Xi} + \tilde{\Xi}\pi_{10}(A)^T - A''\tilde{\Xi}) \otimes M'_N \\
&\quad + \pi_{10,10}(\tilde{\Psi}\tilde{\Phi} - \tfrac{3\mathrm{i}}{5}\tilde{\Phi}) \otimes \tfrac{1}{2}(M'_{10}M'_d + M'_d {M'_{10}}^T) \\
&\quad + (\pi_{10}(\tilde{\Psi})\pi_{10,10}(\tilde{\Phi}) - \pi_{10,10}(\tilde{\Phi})\pi_{10}(\tilde{\Psi})^T + \tfrac{\mathrm{i}}{2}\pi_{10,10}(\tilde{\Upsilon})) \otimes \tfrac{1}{2}(M'_{10}M'_d - M'_d {M'_{10}}^T) \\
&\quad + (\pi_{10}(\tilde{\Psi})\tilde{\Xi} + \tilde{\Xi}\pi_{10}(\tilde{\Psi})^T - \tfrac{12\mathrm{i}}{5}\tilde{\Xi}) \otimes \tfrac{1}{2}(M'_{10}M'_N + M'_N {M'_{10}}^T) \\
&\quad + (\pi_{10}(\tilde{\Psi})\tilde{\Xi} - \tilde{\Xi}\pi_{10}(\tilde{\Psi})^T) \otimes \tfrac{1}{2}(M'_{10}M'_N - M'_N {M'_{10}}^T) \;,
\end{aligned}
\tag{4.7e}
$$

$$
\begin{aligned}
\theta_{10,5} &= -\gamma^5\pi_{10,5}(\mathbf{d}\tilde{\Phi} + (A + A''\mathbb{1}_5)\tilde{\Phi}) \otimes M'_{\tilde{u}} \\
&\quad - \gamma^5(\mathbf{d}\tilde{\Upsilon} + \pi_{10}(A)\tilde{\Upsilon} - \tilde{\Upsilon}\pi_5(A) + A''\tilde{\Upsilon})) \otimes M'_{\tilde{n}} \\
&\quad + (\pi_{10}(\tilde{\Psi})\check{\tilde{\Phi}} - \tfrac{9\mathrm{i}}{20}\check{\tilde{\Phi}} + \tfrac{\mathrm{i}}{4}\tilde{\Upsilon}) \otimes M'_{10}M'_{\tilde{u}} - (\check{\tilde{\Phi}}\pi_5(\tilde{\Psi}) + \tfrac{3\mathrm{i}}{20}\check{\tilde{\Phi}} + \tfrac{\mathrm{i}}{4}\tilde{\Upsilon}) \otimes M'_{\tilde{u}}M'_5 \\
&\quad + (\pi_{10}(\tilde{\Psi})\tilde{\Upsilon} + \tfrac{3\mathrm{i}}{4}\check{\tilde{\Phi}} - \tfrac{19\mathrm{i}}{20}\tilde{\Upsilon}) \otimes M'_{10}M'_{\tilde{n}} - (\tilde{\Upsilon}\pi_5(\tilde{\Psi}) + \tfrac{3\mathrm{i}}{4}\check{\tilde{\Phi}} - \tfrac{7\mathrm{i}}{20}\tilde{\Upsilon}) \otimes M'_{\tilde{n}}M'_5 \;,
\end{aligned}
\tag{4.7f}
$$

$$
\theta_{5,1} = -\gamma^5\pi_{5,1}(\mathbf{d}\tilde{\Phi} + (A + A''\mathbb{1}_5)\tilde{\Phi}) \otimes M'_e + \pi_{5,1}(\tilde{\Psi}\tilde{\Phi} - \tfrac{3\mathrm{i}}{5}\tilde{\Phi}) \otimes {M'_5}^T M'_e \;, \tag{4.7g}
$$

$$
\theta_{10,1} = -\tilde{\Upsilon}\tilde{\Phi} \otimes M'_{\tilde{n}}\bar{M}'_e \;, \tag{4.7h}
$$

$$\theta_{\widetilde{10,5}} = -(\tilde{\Upsilon}^*\hat{\tilde{\Phi}})^T_{\underline{10}} \otimes M'_d \bar{M}'_{\tilde{n}} - (\tilde{\Upsilon}^* \tilde{\Xi})^T_{\underline{10}} \otimes M'_N \bar{M}'_{\tilde{n}} \\ - (\tilde{\Upsilon}^*\hat{\tilde{\Phi}})^T_{\underline{40}} \otimes M'_{d\tilde{n}} - (\tfrac{1}{4}(\tilde{\Upsilon}^* \tilde{\Xi})_{\underline{40}} + \tfrac{3}{4}(\check{\tilde{\Phi}}\tilde{\Xi}))^T \otimes M'_{Nu} \;. \tag{4.7i}$$

4.2. **The Bosonic Action.** It is convenient to put

$$\begin{aligned} &\tilde{\mathbf{\Psi}} := -\mathrm{i}\tilde{\Psi} \;, && \check{\tilde{\mathbf{\Psi}}} := -\mathrm{i}\pi_{10}(\tilde{\Psi}) \;, && \tilde{\mathbf{\Upsilon}} := -\mathrm{i}\tilde{\Upsilon} \;, \\ &\hat{\tilde{\mathbf{\Upsilon}}} := -\mathrm{i}\pi_{10,10}(\tilde{\Upsilon}) \;, && \tilde{\mathbf{\Phi}} := -\mathrm{i}\tilde{\Phi} \;, && \check{\tilde{\mathbf{\Phi}}} := -\mathrm{i}\pi_{10,5}(\tilde{\Phi}) \;, \\ &\hat{\tilde{\mathbf{\Phi}}} := -\mathrm{i}\pi_{10,10}(\tilde{\Phi}) \;, && \tilde{\mathbf{\Xi}} := -\mathrm{i}\tilde{\Xi} \;, && \check{A} := \pi_{10}(A) \;. \end{aligned} \tag{4.8}$$

It turns out that the computation of the bosonic action is not difficult now. The only problem is the length. All what one needs are the orthogonality of different irreducible representations and the relations

$$\begin{aligned} &\mathrm{tr}(\pi_{10}(a)\pi_{10}(\tilde{a})) = 3\,\mathrm{tr}(\pi_5(a)\pi_5(\tilde{a})) = 3\,\mathrm{tr}(a\tilde{a}) \;, \\ &\mathrm{tr}\,\big((A - \tfrac{1}{10}\,\mathrm{tr}(A)\mathbb{1}_{10} - A_{\underline{24}})(\tilde{A} - \tfrac{1}{10}\,\mathrm{tr}(\tilde{A})\mathbb{1}_{10} - \tilde{A}_{\underline{24}})\big) \\ &\qquad\qquad = \mathrm{tr}(A\tilde{A}) - \tfrac{1}{10}\,\mathrm{tr}(A)\;\mathrm{tr}(\tilde{A}) - \mathrm{tr}(A_{\underline{24}}\tilde{A}_{\underline{24}}) \;, \end{aligned} \tag{4.9}$$

for $a, \tilde{a} \in \mathfrak{a}$ and skew–adjoint $A, \tilde{A} \in \mathrm{M}_{10}\mathbb{C}$. We compute the Lagrangian $\mathcal{L} = \frac{1}{192\, g_0^2}\;\mathrm{tr}_c((\mathfrak{e}(\theta))^2)$, where g_0 is a coupling constant and tr_c the combination of the trace over the matrix structure with the trace in the Clifford algebra. For functions $f \in C^\infty(X)$ we have $\mathrm{tr}_c(f) = 4f$. We find:

$$\tfrac{1}{192\, g_0^2}\;\mathrm{tr}_c((\mathfrak{e}(\theta))^2) = \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0 \;, \tag{4.10a}$$

$$\mathcal{L}_2 = \tfrac{1}{4\, g_0^2}\;\mathrm{tr}_c((\mathbf{d}A + \tfrac{1}{2}\{A, A\})^2) + \tfrac{5}{4\, g_0^2}\;\mathrm{tr}_c((\mathbf{d}A'')^2) \;, \tag{4.10b}$$

$$\begin{aligned} \mathcal{L}_1 = {} & \tfrac{1}{g_0^2}\mu_0\;\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{\Psi}} + [A, \tilde{\mathbf{\Psi}}])^2) \\ & + \tfrac{1}{g_0^2}\mu_1\;\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{\Phi}} + (A + A''\mathbb{1}_5)\tilde{\mathbf{\Phi}})^*(\mathbf{d}\tilde{\mathbf{\Phi}} + (A + A''\mathbb{1}_5)\tilde{\mathbf{\Phi}})) \\ & + \tfrac{1}{g_0^2}\mu_2\;\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{\Upsilon}} + \check{A}\tilde{\mathbf{\Upsilon}} - \tilde{\mathbf{\Upsilon}}A + A''\tilde{\mathbf{\Upsilon}})^*(\mathbf{d}\tilde{\mathbf{\Upsilon}} + \check{A}\tilde{\mathbf{\Upsilon}} - \tilde{\mathbf{\Upsilon}}A + A''\tilde{\mathbf{\Upsilon}})) \\ & + \tfrac{1}{g_0^2}\mu_3\;\mathrm{tr}_c((\mathbf{d}\tilde{\mathbf{\Xi}} + \check{A}\tilde{\mathbf{\Xi}} + \tilde{\mathbf{\Xi}}\check{A}^T - A''\tilde{\mathbf{\Xi}})^*(\mathbf{d}\tilde{\mathbf{\Xi}} + \check{A}\tilde{\mathbf{\Xi}} + \tilde{\mathbf{\Xi}}\check{A}^T - A''\tilde{\mathbf{\Xi}}))\;, \end{aligned} \tag{4.10c}$$

$$\begin{aligned} \mathcal{L}_0 = {} & \tfrac{1}{24\, g_0^2}\big\{\mu^{\mathrm{a}}(\mathrm{tr}(\tilde{\mathbf{\Psi}}^2) - \tfrac{6}{5})^2 + \mu^{\mathrm{b}}(\tilde{\mathbf{\Phi}}^*\tilde{\mathbf{\Phi}} - 1)^2 + \mu^{\mathrm{c}}(\mathrm{tr}(\tilde{\mathbf{\Upsilon}}^*\tilde{\mathbf{\Upsilon}}) - 12)^2 \\ & + \mu^{\mathrm{d}}(\mathrm{tr}(\tilde{\mathbf{\Psi}}^2) - \tfrac{6}{5})(\tilde{\mathbf{\Phi}}^*\tilde{\mathbf{\Phi}} - 1) + \mu^{\mathrm{e}}(\mathrm{tr}(\tilde{\mathbf{\Psi}}^2) - \tfrac{6}{5})(\mathrm{tr}(\tilde{\mathbf{\Upsilon}}^*\tilde{\mathbf{\Upsilon}}) - 12) \\ & + \mu^{\mathrm{f}}(\tilde{\mathbf{\Phi}}^*\tilde{\mathbf{\Phi}} - 1)(\mathrm{tr}(\tilde{\mathbf{\Upsilon}}^*\tilde{\mathbf{\Upsilon}}) - 12) \\ & + \check{\mu}^{\mathrm{a}}(\mathrm{tr}(\tilde{\mathbf{\Xi}}\tilde{\mathbf{\Xi}}^*) - 1)^2 + \check{\mu}^{\mathrm{b}}(\mathrm{tr}(\tilde{\mathbf{\Xi}}\tilde{\mathbf{\Xi}}^*) - 1)(\mathrm{tr}(\tilde{\mathbf{\Psi}}^2) - \tfrac{6}{5}) \\ & + \check{\mu}^{\mathrm{c}}(\mathrm{tr}(\tilde{\mathbf{\Xi}}\tilde{\mathbf{\Xi}}^*) - 1)(\tilde{\mathbf{\Phi}}^*\tilde{\mathbf{\Phi}} - 1) + \check{\mu}^{\mathrm{d}}(\mathrm{tr}(\tilde{\mathbf{\Xi}}\tilde{\mathbf{\Xi}}^*) - 1)(\mathrm{tr}(\tilde{\mathbf{\Upsilon}}^*\tilde{\mathbf{\Upsilon}}) - 12) \\ & + \check{\mu}^{\mathrm{e}}\;\mathrm{tr}((\check{\tilde{\mathbf{\Psi}}}\tilde{\mathbf{\Xi}} + \tilde{\mathbf{\Xi}}\check{\tilde{\mathbf{\Psi}}}^T - \tfrac{12}{5}\tilde{\mathbf{\Xi}})(\check{\tilde{\mathbf{\Psi}}}\tilde{\mathbf{\Xi}} + \tilde{\mathbf{\Xi}}\check{\tilde{\mathbf{\Psi}}}^T - \tfrac{12}{5}\tilde{\mathbf{\Xi}})^*) \\ & + \check{\mu}^{\mathrm{f}}\;\mathrm{tr}((\check{\tilde{\mathbf{\Psi}}}\tilde{\mathbf{\Xi}} - \tilde{\mathbf{\Xi}}\check{\tilde{\mathbf{\Psi}}}^T)(\check{\tilde{\mathbf{\Psi}}}\tilde{\mathbf{\Xi}} - \tilde{\mathbf{\Xi}}\check{\tilde{\mathbf{\Psi}}}^T)^*) \\ & + \check{\mu}^{\mathrm{g}}\,\mathrm{Re}(\mathrm{tr}((\check{\tilde{\mathbf{\Psi}}}\hat{\tilde{\mathbf{\Phi}}} - \hat{\tilde{\mathbf{\Phi}}}\check{\tilde{\mathbf{\Psi}}}^T + \tfrac{1}{2}\hat{\tilde{\mathbf{\Upsilon}}})(\check{\tilde{\mathbf{\Psi}}}\tilde{\mathbf{\Xi}} - \tilde{\mathbf{\Xi}}\check{\tilde{\mathbf{\Psi}}}^T)^*)) \\ & + \check{\mu}^{\mathrm{h}}\,\mathrm{Im}(\mathrm{tr}((\check{\tilde{\mathbf{\Psi}}}\hat{\tilde{\mathbf{\Phi}}} - \hat{\tilde{\mathbf{\Phi}}}\check{\tilde{\mathbf{\Psi}}}^T + \tfrac{1}{2}\hat{\tilde{\mathbf{\Upsilon}}})(\check{\tilde{\mathbf{\Psi}}}\tilde{\mathbf{\Xi}} - \tilde{\mathbf{\Xi}}\check{\tilde{\mathbf{\Psi}}}^T)^*)) \\ & + \mu^{\mathrm{g}}\,\tilde{\mathbf{\Phi}}^*(\tilde{\mathbf{\Psi}} - \tfrac{3}{5}\mathbb{1}_5)^2\tilde{\mathbf{\Phi}} + \mu^{\mathrm{h}}\;\mathrm{tr}((\check{\tilde{\mathbf{\Psi}}}\hat{\tilde{\mathbf{\Phi}}} - \hat{\tilde{\mathbf{\Phi}}}\check{\tilde{\mathbf{\Psi}}}^T + \tfrac{1}{2}\hat{\tilde{\mathbf{\Upsilon}}})(\check{\tilde{\mathbf{\Psi}}}\hat{\tilde{\mathbf{\Phi}}} - \hat{\tilde{\mathbf{\Phi}}}\check{\tilde{\mathbf{\Psi}}}^T + \tfrac{1}{2}\hat{\tilde{\mathbf{\Upsilon}}})^*) \\ & + \mu^{\mathrm{i}}\;\mathrm{tr}((\check{\tilde{\mathbf{\Psi}}}\check{\tilde{\mathbf{\Phi}}} - \tfrac{9}{20}\check{\tilde{\mathbf{\Phi}}} + \tfrac{1}{4}\tilde{\mathbf{\Upsilon}})^*(\check{\tilde{\mathbf{\Psi}}}\check{\tilde{\mathbf{\Phi}}} - \tfrac{9}{20}\check{\tilde{\mathbf{\Phi}}} + \tfrac{1}{4}\tilde{\mathbf{\Upsilon}})) \\ & + \mu^{\mathrm{j}}\;\mathrm{tr}((\check{\tilde{\mathbf{\Phi}}}\tilde{\mathbf{\Psi}} + \tfrac{3}{20}\check{\tilde{\mathbf{\Phi}}} + \tfrac{1}{4}\tilde{\mathbf{\Upsilon}})^*(\check{\tilde{\mathbf{\Phi}}}\tilde{\mathbf{\Psi}} + \tfrac{3}{20}\check{\tilde{\mathbf{\Phi}}} + \tfrac{1}{4}\tilde{\mathbf{\Upsilon}})) \end{aligned} \tag{4.10d}$$

$$
\begin{aligned}
&+ \mu^{\mathrm{k}}\ \mathrm{tr}\big((\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})\big)\\
&+ \mu^{\mathrm{l}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\big)\\
&- \mu^{\mathrm{m}}\,\mathrm{Re}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&+ \mu^{\mathrm{n}}\,\mathrm{Re}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&+ \mu^{\mathrm{o}}\,\mathrm{Im}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&- \mu^{\mathrm{p}}\,\mathrm{Re}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\\
&\qquad\qquad + (\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&- \mu^{\mathrm{q}}\,\mathrm{Im}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Psi}}}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{9}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\\
&\qquad\qquad + (\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&+ \mu^{\mathrm{r}}\,\mathrm{Re}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&+ \mu^{\mathrm{s}}\,\mathrm{Im}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{20}\check{\tilde{\boldsymbol{\Phi}}} + \tfrac{1}{4}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&- \mu^{\mathrm{t}}\,\mathrm{Re}\big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Upsilon}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{19}{20}\tilde{\boldsymbol{\Upsilon}})^*(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Psi}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}} - \tfrac{7}{20}\tilde{\boldsymbol{\Upsilon}})\big)\big)\\
&+ \mu^{\mathrm{u}}\,\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Phi}} + \mu^{\mathrm{v}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{10}}\big) + \mu^{\mathrm{w}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{40}}(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{40}}\big)\\
&+ \check{\mu}^{\mathrm{i}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})^*_{\underline{10}}\big) + \check{\mu}^{\mathrm{l}}\ \mathrm{tr}\big((\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})(\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})^*\big)\\
&+ \check{\mu}^{\mathrm{j}}\,\mathrm{Re}\big(\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})^*_{\underline{10}}\big)\big) + \check{\mu}^{\mathrm{m}}\,\mathrm{Re}\big(\mathrm{tr}\big((\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{40}}\big)\big)\\
&+ \check{\mu}^{\mathrm{k}}\,\mathrm{Im}\big(\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})_{\underline{10}}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})^*_{\underline{10}}\big)\big) + \check{\mu}^{\mathrm{n}}\,\mathrm{Im}\big(\mathrm{tr}\big((\tfrac{1}{4}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Xi}})_{\underline{40}} + \tfrac{3}{4}\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}})(\tilde{\boldsymbol{\Upsilon}}^*\hat{\tilde{\boldsymbol{\Phi}}})^*_{\underline{40}}\big)\big)\\
&+ \tilde{\mu}^{\mathrm{a}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Psi}}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5 - \tfrac{1}{5}\tilde{\boldsymbol{\Psi}})^2\big)\\
&+ \tilde{\mu}^{\mathrm{b}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5 - 8\tilde{\boldsymbol{\Psi}} + 9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* - \tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)^2\big)\\
&+ \tilde{\mu}^{\mathrm{c}}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)' - \tfrac{8}{3}\tilde{\boldsymbol{\Psi}} - \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{1}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)^2\big)\\
&+ \tilde{\mu}^{\mathrm{d}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}} + 8\tilde{\boldsymbol{\Psi}} - 6\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{6}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)^2\big) + \tilde{\mu}^{\mathrm{e}}\ \mathrm{tr}\big(-(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})^2\big)\\
&+ \tilde{\mu}^{\mathrm{f}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Psi}}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5 - \tfrac{1}{5}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5 - 8\tilde{\boldsymbol{\Psi}} + 9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* - \tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+ \tilde{\mu}^{\mathrm{g}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Psi}}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5 - \tfrac{1}{5}\tilde{\boldsymbol{\Psi}})((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)' - \tfrac{8}{3}\tilde{\boldsymbol{\Psi}} - \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{1}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+ \tilde{\mu}^{\mathrm{h}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Psi}}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5 - \tfrac{1}{5}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}} + 8\tilde{\boldsymbol{\Psi}} - 6\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{6}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+ \tilde{\mu}^{\mathrm{i}}\,\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Psi}}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5 - \tfrac{1}{5}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\\
&+ \tilde{\mu}^{\mathrm{j}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5 - 8\tilde{\boldsymbol{\Psi}} + 9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* - \tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\times\\
&\qquad \times((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)' - \tfrac{8}{3}\tilde{\boldsymbol{\Psi}} - \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{1}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+ \tilde{\mu}^{\mathrm{k}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5 - 8\tilde{\boldsymbol{\Psi}} + 9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* - \tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\times\\
&\qquad \times(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}} + 8\tilde{\boldsymbol{\Psi}} - 6\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{6}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+ \tilde{\mu}^{\mathrm{l}}\,\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5 - 8\tilde{\boldsymbol{\Psi}} + 9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* - \tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\\
&+ \tilde{\mu}^{\mathrm{m}}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)' - \tfrac{8}{3}\tilde{\boldsymbol{\Psi}} - \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{1}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\times\\
&\qquad\qquad \times(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}} + 8\tilde{\boldsymbol{\Psi}} - 6\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{6}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+ \tilde{\mu}^{\mathrm{n}}\,\mathrm{i}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)' - \tfrac{8}{3}\tilde{\boldsymbol{\Psi}} - \tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{1}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\\
&+ \tilde{\mu}^{\mathrm{o}}\,\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} + \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}} + 8\tilde{\boldsymbol{\Psi}} - 6\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* + \tfrac{6}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}} - \check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\\
&+ \hat{\mu}^{\mathrm{a}}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)' - \tfrac{1}{3}\tilde{\boldsymbol{\Psi}})^2\big) + \hat{\mu}^{\mathrm{b}}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)' - \tfrac{1}{3}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Psi}}^2 - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5 - \tfrac{1}{5}\tilde{\boldsymbol{\Psi}})\big)\\
&+ \hat{\mu}^{\mathrm{c}}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)' - \tfrac{1}{3}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}} - \tfrac{1}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}})\mathbb{1}_5 - 8\tilde{\boldsymbol{\Psi}} + 9\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^* - \tfrac{9}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\hat{\mu}^{\mathrm{d}}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)'-\tfrac{1}{3}\tilde{\boldsymbol{\Psi}})((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)'-\tfrac{8}{3}\tilde{\boldsymbol{\Psi}}-\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^*+\tfrac{1}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+\hat{\mu}^{\mathrm{e}}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)'-\tfrac{1}{3}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}+\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}}+8\tilde{\boldsymbol{\Psi}}-6\tilde{\boldsymbol{\Phi}}\tilde{\boldsymbol{\Phi}}^*+\tfrac{6}{5}\tilde{\boldsymbol{\Phi}}^*\tilde{\boldsymbol{\Phi}}\mathbb{1}_5)\big)\\
&+\hat{\mu}^{\mathrm{f}}\,\mathrm{i}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)'-\tfrac{1}{3}\tilde{\boldsymbol{\Psi}})(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}-\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\\
&+\tilde{\mu}^{\mathrm{p}}\Big(\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*-4\check{\tilde{\boldsymbol{\Psi}}}^2)^2\big)-\tfrac{1}{10}\big(\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}}-12\tilde{\boldsymbol{\Psi}}^2)\big)^2-\\
&\qquad\qquad -3\,\mathrm{tr}\big(((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)'-\tfrac{4}{3}\tilde{\boldsymbol{\Psi}}^2+\tfrac{4}{15}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)^2\big)\Big)\\
&+\tilde{\mu}^{\mathrm{q}}\Big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*+4\check{\tilde{\boldsymbol{\Psi}}}^2)^2\big)-\tfrac{72}{5}\big(\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\big)^2-\\
&\qquad\qquad -\tfrac{1}{3}\,\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}+\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}}+4\tilde{\boldsymbol{\Psi}}^2-\tfrac{4}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)^2\big)\Big)\\
&+\tilde{\mu}^{\mathrm{r}}\Big(\mathrm{tr}\big(-(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*)^2\big)+\tfrac{1}{3}\,\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}-\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})^2\big)\Big)\\
&+\tilde{\mu}^{\mathrm{s}}\Big(\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*-4\check{\tilde{\boldsymbol{\Psi}}}^2)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*+4\check{\tilde{\boldsymbol{\Psi}}}^2)\big)-\tfrac{6}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}}-12\tilde{\boldsymbol{\Psi}}^2)-\\
&\qquad -\mathrm{tr}\big(((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)'-\tfrac{4}{3}\tilde{\boldsymbol{\Psi}}^2+\tfrac{4}{15}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}+\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}}+4\tilde{\boldsymbol{\Psi}}^2-\tfrac{4}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)\big)\Big)\\
&+\tilde{\mu}^{\mathrm{t}}\,\mathrm{i}\Big(\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*-4\check{\tilde{\boldsymbol{\Psi}}}^2)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*)\big)-\mathrm{tr}\big(((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)'-\tfrac{4}{3}\tilde{\boldsymbol{\Psi}}^2)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}-\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\Big)\\
&+\tilde{\mu}^{\mathrm{u}}\,\mathrm{i}\Big(\mathrm{tr}\big((\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*+4\check{\tilde{\boldsymbol{\Psi}}}^2)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*)\big)-\\
&\qquad\qquad -\tfrac{1}{3}\,\mathrm{tr}\big((\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}+\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}}+4\tilde{\boldsymbol{\Psi}}^2)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}-\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\Big)\\
&+\hat{\mu}^{\mathrm{g}}\Big(\mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^2)^2\big)-\tfrac{1}{10}\big(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)-\tfrac{3}{2}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\big)^2-\\
&\qquad\qquad -3\,\mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)'-\tfrac{1}{6}\tilde{\boldsymbol{\Psi}}^2+\tfrac{1}{30}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)^2\big)\Big)\\
&+\hat{\mu}^{\mathrm{h}}\,\mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)^2\big)-\hat{\mu}^{\mathrm{i}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)^2\big)\\
&+\hat{\mu}^{\mathrm{j}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^2)(\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)\big)\\
&+\hat{\mu}^{\mathrm{k}}\,\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^2)(\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)\big)+\hat{\mu}^{\mathrm{l}}\,\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)(\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)\big)\\
&+\hat{\mu}^{\mathrm{m}}\Big(\mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^2)(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*-4\check{\tilde{\boldsymbol{\Psi}}}^2)\big)-\tfrac{1}{10}\big(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)-\tfrac{3}{2}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\big)\,\mathrm{tr}(\tilde{\boldsymbol{\Upsilon}}^*\tilde{\boldsymbol{\Upsilon}}-12\tilde{\boldsymbol{\Psi}}^2)-\\
&\qquad\qquad -3\,\mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)'-\tfrac{1}{6}\tilde{\boldsymbol{\Psi}}^2+\tfrac{1}{30}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)((\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*)'-\tfrac{4}{3}\tilde{\boldsymbol{\Psi}}^2+\tfrac{4}{15}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)\big)\Big)\\
&+\hat{\mu}^{\mathrm{n}}\Big(\mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^2)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*+4\check{\tilde{\boldsymbol{\Psi}}}^2)\big)-\tfrac{12}{10}\big(\mathrm{tr}(\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)-\tfrac{3}{2}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\big)\ \mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)-\\
&\qquad\quad -\mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)'-\tfrac{1}{6}\tilde{\boldsymbol{\Psi}}^2+\tfrac{1}{30}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}+\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}}+4\tilde{\boldsymbol{\Psi}}^2-\tfrac{4}{5}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)\big)\Big)\\
&+\hat{\mu}^{\mathrm{o}}\Big(\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*-\tfrac{1}{2}\check{\tilde{\boldsymbol{\Psi}}}^2)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*)\big)-\\
&\qquad\qquad -\mathrm{i}\ \mathrm{tr}\big(((\tilde{\boldsymbol{\Xi}}\tilde{\boldsymbol{\Xi}}^*)'-\tfrac{1}{6}\tilde{\boldsymbol{\Psi}}^2+\tfrac{1}{30}\,\mathrm{tr}(\tilde{\boldsymbol{\Psi}}^2)\mathbb{1}_5)(\tilde{\boldsymbol{\Upsilon}}^*\check{\tilde{\boldsymbol{\Phi}}}-\check{\tilde{\boldsymbol{\Phi}}}^*\tilde{\boldsymbol{\Upsilon}})\big)\Big)\\
&+\hat{\mu}^{\mathrm{p}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*-4\check{\tilde{\boldsymbol{\Psi}}}^2)\big)\\
&+\hat{\mu}^{\mathrm{q}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*+4\check{\tilde{\boldsymbol{\Psi}}}^2)\big)\\
&+\hat{\mu}^{\mathrm{r}}\,\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*+\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*)\big)+\hat{\mu}^{\mathrm{s}}\,\mathrm{i}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)(\tilde{\boldsymbol{\Upsilon}}\tilde{\boldsymbol{\Upsilon}}^*-4\check{\tilde{\boldsymbol{\Psi}}}^2)\big)\\
&+\hat{\mu}^{\mathrm{t}}\,\mathrm{i}\,\mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*+\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*+4\check{\tilde{\boldsymbol{\Psi}}}^2)\big)\\
&-\hat{\mu}^{\mathrm{u}}\ \mathrm{tr}\big((\tilde{\boldsymbol{\Xi}}\hat{\tilde{\boldsymbol{\Phi}}}^*-\hat{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Xi}}^*)(\check{\tilde{\boldsymbol{\Phi}}}\tilde{\boldsymbol{\Upsilon}}^*-\tilde{\boldsymbol{\Upsilon}}\check{\tilde{\boldsymbol{\Phi}}}^*)\big)\Big\}\,,
\end{aligned}
$$

where the coefficients μ^i are given in Appendix B.

The group of local gauge transformations associated to our model is

$$
\mathcal{U}_0(\mathfrak{g})=\exp(\pi(C^\infty(X)\otimes(\mathrm{su}(5)\oplus\mathrm{u}(1))))\cong C^\infty(X)\otimes(\mathrm{SU}(5)\times\mathrm{U}(1))\,. \qquad (4.11)
$$

The Lagrangian (4.10) is invariant under the gauge transformations

$$\begin{aligned}
\gamma_u(A) &= u_5 \mathbf{d} u_5^* + u_5 A u_5^* , & \gamma_u(\check{A}) &= u_{10} \mathbf{d} u_{10}^* + u_{10} \check{A} u_{10}^* , \\
\gamma_u(A'') &= u_1 \mathbf{d} u_1^* + A'' , \\
\gamma_u(\tilde{\mathbf{\Upsilon}}) &= u_1 u_{10} \tilde{\mathbf{\Upsilon}} u_5^* , & \gamma_u(\hat{\tilde{\mathbf{\Upsilon}}}) &= u_1^* u_{10} \hat{\tilde{\mathbf{\Upsilon}}} u_{10}^T , \\
\gamma_u(\tilde{\mathbf{\Psi}}) &= u_5 \tilde{\mathbf{\Psi}} u_5^* , & \gamma_u(\check{\tilde{\mathbf{\Psi}}}) &= u_{10} \check{\tilde{\mathbf{\Psi}}} u_{10}^* , \\
\gamma_u(\tilde{\mathbf{\Phi}}) &= u_1 u_5 \tilde{\mathbf{\Phi}} , & \gamma_u(\check{\tilde{\mathbf{\Phi}}}) &= u_1 u_{10} \check{\tilde{\mathbf{\Phi}}} u_5^* , \\
\gamma_u(\hat{\tilde{\mathbf{\Phi}}}) &= u_1^* u_{10} \hat{\tilde{\mathbf{\Phi}}} u_{10}^T , & \gamma_u(\tilde{\mathbf{\Xi}}) &= u_1^* u_{10} \tilde{\mathbf{\Xi}} u_{10}^T ,
\end{aligned} \tag{4.12a}$$

where

$$\begin{aligned}
&u_5 = \exp(t_5) , \qquad u_{10} = \exp(\pi_{10}(t_5)) , \qquad && t_5 \in C^\infty(X) \otimes \mathrm{su}(5) , \\
&u_1 = \exp(t_1) , && t_1 \in C^\infty(X) \otimes \mathrm{u}(1) .
\end{aligned} \tag{4.12b}$$

To determine the spontaneous symmetry breaking pattern, we must search for a local minimum of the Higgs potential $\mathcal{L}_0$. This problem is easy to solve. We know that, applying the transformation (4.4b) in the other direction, the Λ^0–part of the curvature $\mathfrak{e}(\theta)$ (and hence the Higgs potential $\mathcal{L}_0$) is zero for

$$\begin{aligned}
&\Psi = 0 , \quad \Phi = 0 , \quad \Xi = 0 , \quad \Upsilon = 0 \quad \text{or} \\
&\tilde{\Psi} = m , \quad \tilde{\Phi} = n , \quad \tilde{\Xi} = m' , \quad \tilde{\Upsilon} = n' .
\end{aligned} \tag{4.13}$$

Since the Higgs potential $\mathcal{L}_0$ is not negative as the trace of the square of the Λ^0–part of the selfadjoint matrix $\mathfrak{e}(\theta)$, the point (4.13) is a global minimum of $\mathcal{L}_0$. But (4.13) is clearly a local minimum as well: In the vicinity of (4.13), the Λ^0–part of $\mathfrak{e}(\theta)$ is linear in the components of Ψ, Φ, Ξ and Υ so that the Higgs potential $\mathcal{L}_0$ is in lowest order quadratic in these components.

We underline that, given the fermion masses and the spontaneous symmetry breaking pattern as the input, our formalism provides a straightforward algorithm to determine the occurring Higgs multiplets and their most general gauge invariant Higgs potential.

4.3. **The Bosonic Lagrangian in Local Coordinates.** In this subsection we will write down the Lagrangian (4.10) in terms of local coordinates. We must restrict ourselves concerning the Higgs potential (4.10d) to the terms quadratic in the fields, because the complete expansion of $\mathcal{L}_0$ is too voluminous. Let us introduce in the same way as in (4.8) the bold matrices

$$\begin{aligned}
&\mathbf{m} := -\mathrm{i}\pi_5(m) \equiv \mathrm{diag}(-\tfrac{2}{5}, -\tfrac{2}{5}, -\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{5}) , \\
&\check{\mathbf{m}} := -\mathrm{i}\pi_{10}(m) \equiv \mathrm{diag}(\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, -\tfrac{4}{5}, -\tfrac{4}{5}, -\tfrac{4}{5}, \tfrac{6}{5}) , \\
&\mathbf{n}' := -\mathrm{i}n' \equiv \begin{pmatrix} \mathbb{1}_3 & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times1} & 3 \end{pmatrix} , \quad \check{\mathbf{n}} := -\mathrm{i}\pi_{10,5}(n) \equiv \begin{pmatrix} \mathbb{1}_3 & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times1} & -1 \end{pmatrix} ,
\end{aligned} \tag{4.14}$$

$$\mathbf{n} := \overline{-\mathrm{i}\pi_{5,1}(n)} \equiv \begin{pmatrix} 0_{3\times1} \\ 1 \\ 0_{1\times1} \end{pmatrix} \;,\; \hat{\mathbf{n}} := -\mathrm{i}\pi_{10,10}(n) \equiv \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & -\mathbb{1}_3 & 0_{3\times1} \\ 0_{3\times3} & -\mathbb{1}_3 & 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times1} \end{pmatrix} ,$$

$$\mathbf{m}' := -\mathrm{i}m' \equiv \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & -1 \end{pmatrix} ,$$

$$\hat{\mathbf{n}}' := -\mathrm{i}\pi_{10,10}(n') \equiv \begin{pmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times1} \\ 0_{3\times3} & 0_{3\times3} & 2\mathbb{1}_3 & 0_{3\times1} \\ 0_{3\times3} & -2\mathbb{1}_3 & 0_{3\times3} & 0_{3\times1} \\ 0_{1\times3} & 0_{1\times3} & 0_{1\times3} & 0_{1\times1} \end{pmatrix} ,$$

see (3.12). We shall write our formulae in terms of the "physical" fields $\mathbf{\Psi}, \mathbf{\Phi}, \mathbf{\Xi}, \mathbf{\Upsilon}$ given by

$$\tilde{\mathbf{\Psi}} = \mathbf{\Psi} + \mathbf{m} \;, \qquad \tilde{\mathbf{\Phi}} = \mathbf{\Phi} + \mathbf{n} \;, \qquad \tilde{\mathbf{\Xi}} = \mathbf{\Xi} + \mathbf{m}' \;, \qquad \tilde{\mathbf{\Upsilon}} = \mathbf{\Upsilon} + \mathbf{n}' \;. \tag{4.15}$$

The subgroup of $C^\infty(X)\otimes(\mathrm{SU}(5)\times\mathrm{U}(1))$, which leaves (4.13) invariant, is the group $C^\infty(X)\otimes(\mathrm{SU}(3)_C\times\mathrm{U}(1)_{EM})$. The Higgs mechanism consists in reducing the symmetry of the whole theory to the symmetry of the vacuum. This means that we fix the gauge transformations corresponding to

$$C^\infty(X)\otimes\big((\mathrm{SU}(5)\times\mathrm{U}(1))\,/\,(\mathrm{SU}(3)_C\times\mathrm{U}(1)_{EM})\big)$$

in such a way that the Higgs multiplets $\mathbf{\Psi}$, $\mathbf{\Phi}$ and $\mathbf{\Xi}$ take the form

$$\mathbf{\Psi} = \begin{pmatrix} -\sqrt{\frac{4}{15}}\varPsi_0\mathbb{1}_3 + \mathbf{\Psi}_g & 0 \\ 0 & \sqrt{\frac{3}{5}}\varPsi_0 + \mathbf{\Psi}_w \end{pmatrix} , \tag{4.16a}$$

$$\mathbf{\Psi}_g = \begin{pmatrix} \sqrt{\frac{1}{3}}\varPsi_8+\varPsi_3 & \varPsi_1-\mathrm{i}\varPsi_2 & \varPsi_4-\mathrm{i}\varPsi_5 \\ \varPsi_1+\mathrm{i}\varPsi_2 & \sqrt{\frac{1}{3}}\varPsi_8-\varPsi_3 & \varPsi_6-\mathrm{i}\varPsi_7 \\ \varPsi_4+\mathrm{i}\varPsi_5 & \varPsi_6+\mathrm{i}\varPsi_7 & -\sqrt{\frac{4}{3}}\varPsi_8 \end{pmatrix} = \sum_{a=1}^{8}\varPsi_a\lambda^a \;,\; \varPsi_a\in C^\infty(X) \;, \tag{4.16b}$$

$$\mathbf{\Psi}_w = \begin{pmatrix} \varPsi_3' & \varPsi_1' - \mathrm{i}\varPsi_2' \\ \varPsi_1' + \mathrm{i}\varPsi_2' & -\varPsi_3' \end{pmatrix} = \sum_{a=1}^{3}\varPsi_a'\sigma^a \;, \quad \varPsi_a'\in C^\infty(X) \;, \tag{4.16c}$$

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{\Phi}_g \\ \mathbf{\Phi}_w \end{pmatrix} \;,\; \mathbf{\Phi}_g = \begin{pmatrix} \varPhi_1 + \mathrm{i}\varPhi_4 \\ \varPhi_2 + \mathrm{i}\varPhi_5 \\ \varPhi_3 + \mathrm{i}\varPhi_6 \end{pmatrix} \;,\; \mathbf{\Phi}_w = \begin{pmatrix} \varPhi_0 \\ 0 \end{pmatrix} \;, \quad \varPhi_a\in C^\infty(X) \;, \tag{4.16d}$$

$$\mathbf{\Xi} = \begin{pmatrix} \overline{\mathbf{\Xi}_A} & \overline{\mathbf{\Xi}_D} - \frac{1}{2}\varepsilon(\overline{\mathbf{\Xi}_c}) & (\mathbf{\Xi}_E^0)^* & \mathbf{\Xi}_a \\ \overline{\mathbf{\Xi}_D} + \frac{1}{2}\varepsilon(\overline{\mathbf{\Xi}_c}) & \overline{\mathbf{\Xi}_B} & (\mathbf{\Xi}_F^0)^* & \mathbf{\Xi}_b \\ \overline{\mathbf{\Xi}_E^0} & \overline{\mathbf{\Xi}_F^0} & \mathbf{\Xi}_C & \overline{\mathbf{\Xi}_c} \\ \mathbf{\Xi}_a^T & \mathbf{\Xi}_b^T & \mathbf{\Xi}_c^* & -\mathbf{\Xi}_0 \end{pmatrix} , \tag{4.16e}$$

where $\mathbf{\Xi}_0 \in C^\infty(X)$ is a *real* function. The explicit form of $\mathbf{\Xi}$ is presented in (4.17), where $\Xi_i \in C^\infty(X)$, $i = 0, \dots, 98$. Here, λ^a are the Gell–Mann matrices and σ^a the Pauli matrices. The matrix $\mathbf{\Upsilon}$ is an arbitrary element of $\mathrm{i}\mathfrak{w}$ as displayed in (4.18), where $\Upsilon_i \in C^\infty(X)$.

For A and A'' we make the ansatz

$$A = \frac{\mathrm{i}g_0}{2} \begin{pmatrix} \sqrt{\frac{4}{15}} A' \mathbb{1}_3 + \mathbf{G} & \mathbf{X} \\ \mathbf{X}^* & -\sqrt{\frac{3}{5}} A' \mathbb{1}_2 + \mathbf{W} \end{pmatrix} , \quad A' \in \Lambda^1 , \tag{4.19a}$$

$$A'' = \tfrac{\mathrm{i}g_0}{2} \sqrt{\tfrac{2}{5}} \tilde{A} , \quad \tilde{A} \in \Lambda^1 , \tag{4.19b}$$

$$\mathbf{G} = \begin{pmatrix} \sqrt{\frac{1}{3}} G^8 + G^3 & G^1 - \mathrm{i}G^2 & G^4 - \mathrm{i}G^5 \\ G^1 + \mathrm{i}G^2 & \sqrt{\frac{1}{3}} G^8 - G^3 & G^6 - \mathrm{i}G^7 \\ G^4 + \mathrm{i}G^5 & G^6 + \mathrm{i}G^7 & -\sqrt{\frac{4}{3}} G^8 \end{pmatrix} = \sum_{a=1}^{8} G^a \lambda^a , \quad G^a \in \Lambda^1 , \tag{4.19c}$$

$$\mathbf{W} = \begin{pmatrix} W^3 & W^1 - \mathrm{i}W^2 \\ W^1 + \mathrm{i}W^2 & -W^3 \end{pmatrix} = \sum_{a=1}^{3} W^a \sigma^a , \quad W^a \in \Lambda^1 , \tag{4.19d}$$

$$\mathbf{X} = \begin{pmatrix} X , Y \end{pmatrix} , \quad X = \begin{pmatrix} X^1 - \mathrm{i}X^2 \\ X^3 - \mathrm{i}X^4 \\ X^5 - \mathrm{i}X^6 \end{pmatrix} , \quad Y = \begin{pmatrix} Y^1 - \mathrm{i}Y^2 \\ Y^3 - \mathrm{i}Y^4 \\ Y^5 - \mathrm{i}Y^6 \end{pmatrix} , \quad X^a, Y^a \in \Lambda^1 . \tag{4.19e}$$

In terms of the local basis $\{\gamma^\mu\}_{\mu=1,2,3,4}$ of Λ^1 we put

$$\begin{aligned} &\mathbf{G} = \mathbf{G}_\mu \gamma^\mu , \quad G^a = G^a_\mu \gamma^\mu , \quad W^a = W^a_\mu \gamma^\mu , \quad A' = A'_\mu \gamma^\mu , \quad \tilde{A} = \tilde{A}_\mu \gamma^\mu , \\ &\mathbf{X} = \mathbf{X}_\mu \gamma^\mu , \quad X = X_\mu \gamma^\mu , \quad X^a = X^a_\mu \gamma^\mu , \quad Y = Y_\mu \gamma^\mu , \quad Y^a = Y^a_\mu \gamma^\mu . \end{aligned}$$

Moreover, we introduce the abbreviation

$$S_{[\mu} T_{\nu]} := S_\mu T_\nu - S_\nu T_\mu .$$

Now we start to write down the explicit form of the Lagrangian $\mathcal{L}_2$, where we restrict ourselves to the interesting part and denote the rest by $I.T$ (interaction terms). We obtain in terms of the local basis $\gamma^\mu \wedge \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ of Λ^2

$$\mathbf{d}A + \tfrac{1}{2}\{A, A\} = \frac{\mathrm{i}g_0}{4} \begin{pmatrix} \begin{bmatrix} \frac{2}{3}(\sqrt{\frac{3}{5}} A'_{\mu\nu} - X^0_{\mu\nu}) \mathbb{1}_3 \\ + \sum_{a=1}^{8} (G^a_{\mu\nu} - X^a_{\mu\nu}) \lambda^a \end{bmatrix} & (D\mathbf{X})_{\mu\nu} \\ (D\mathbf{X})^*_{\mu\nu} & \begin{bmatrix} (-\sqrt{\frac{3}{5}} A'_{\mu\nu} + X^0_{\mu\nu}) \mathbb{1}_2 \\ + \sum_{a=1}^{3} (W^a_{\mu\nu} - \tilde{X}^a_{\mu\nu}) \sigma^a \end{bmatrix} \end{pmatrix} \gamma^\mu \wedge \gamma^\nu , \tag{4.20a}$$

where

$$\begin{aligned} &G^a_{\mu\nu} = \partial_{[\mu} G^a_{\nu]} - g_0 \textstyle\sum_{b,c=1}^{8} f_{abc} G^b_\mu G^c_\nu , \qquad W^a_{\mu\nu} = \partial_{[\mu} W^a_{\nu]} - g_0 \textstyle\sum_{b,c=1}^{3} \varepsilon_{abc} W^b_\mu W^c_\nu , \\ &A'_{\mu\nu} = \partial_{[\mu} A'_{\nu]} , \qquad (D\mathbf{X})_{\mu\nu} = \big(\partial_{[\mu} X_{\nu]} + I.T , \ \partial_{[\mu} Y_{\nu]} + I.T \big) , \end{aligned} \tag{4.20b}$$

and $X^a_{\mu\nu}, \tilde{X}^{\tilde{a}}_{\mu\nu}$ are interaction terms. Moreover,

$$\mathbf{d}A'' = \mathrm{i}\tfrac{g_0}{4} \sqrt{\tfrac{2}{5}} \tilde{A}_{\mu\nu} , \qquad \tilde{A}_{\mu\nu} := \partial_{[\mu} \tilde{A}_{\nu]} . \tag{4.21}$$

$$\mathbf{\Xi} = \tag{4.17}$$

$$\left(\begin{array}{ccc|ccc|ccc|c}
\begin{bmatrix}\frac{1}{\sqrt3}(\Xi_1-\mathrm{i}\Xi_{50})\\+\frac{1}{\sqrt2}(\Xi_2-\mathrm{i}\Xi_{51})\\+\frac{1}{\sqrt6}(\Xi_3-\mathrm{i}\Xi_{52})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_4-\mathrm{i}\Xi_{53}) & \frac{1}{\sqrt2}(\Xi_5-\mathrm{i}\Xi_{54}) & \begin{bmatrix}\frac{1}{\sqrt6}(\Xi_{19}-\mathrm{i}\Xi_{68})\\+\frac12(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{22}-\mathrm{i}\Xi_{71})\\-\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{23}-\mathrm{i}\Xi_{72})\\+\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{25}-\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}-\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{28}-\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}-\mathrm{i}\Xi_{78})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{41}+\mathrm{i}\Xi_{90}) \\
\frac{1}{\sqrt2}(\Xi_4-\mathrm{i}\Xi_{53}) & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_1-\mathrm{i}\Xi_{50})\\-\frac{1}{\sqrt2}(\Xi_2-\mathrm{i}\Xi_{51})\\+\frac{1}{\sqrt6}(\Xi_3-\mathrm{i}\Xi_{52})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_6-\mathrm{i}\Xi_{55}) & \begin{bmatrix}\frac12(\Xi_{22}-\mathrm{i}\Xi_{71})\\+\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt6}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac12(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{24}-\mathrm{i}\Xi_{73})\\-\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{25}+\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}+\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}-\frac12(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{30}-\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}-\mathrm{i}\Xi_{80})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{42}+\mathrm{i}\Xi_{91}) \\
\frac{1}{\sqrt2}(\Xi_5-\mathrm{i}\Xi_{54}) & \frac{1}{\sqrt2}(\Xi_6-\mathrm{i}\Xi_{55}) & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_1-\mathrm{i}\Xi_{50})\\-\frac{2}{\sqrt6}(\Xi_3-\mathrm{i}\Xi_{52})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{23}-\mathrm{i}\Xi_{72})\\-\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{24}-\mathrm{i}\Xi_{73})\\+\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt6}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac{2}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{28}+\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}+\mathrm{i}\Xi_{78})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{30}+\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}+\mathrm{i}\Xi_{80})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81}) & \frac{1}{\sqrt2}(\Xi_{43}+\mathrm{i}\Xi_{92}) \\
\hline
\begin{bmatrix}\frac{1}{\sqrt6}(\Xi_{19}-\mathrm{i}\Xi_{68})\\+\frac12(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{22}-\mathrm{i}\Xi_{71})\\+\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{23}-\mathrm{i}\Xi_{72})\\-\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_7-\mathrm{i}\Xi_{56})\\+\frac{1}{\sqrt2}(\Xi_8-\mathrm{i}\Xi_{57})\\+\frac{1}{\sqrt6}(\Xi_9-\mathrm{i}\Xi_{58})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{10}-\mathrm{i}\Xi_{59}) & \frac{1}{\sqrt2}(\Xi_{11}-\mathrm{i}\Xi_{60}) & \begin{bmatrix}\frac12(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{33}-\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}-\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{36}-\mathrm{i}\Xi_{37})\\-\frac{\mathrm{i}}{2}(\Xi_{85}-\mathrm{i}\Xi_{86})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{44}+\mathrm{i}\Xi_{93}) \\
\begin{bmatrix}\frac12(\Xi_{22}-\mathrm{i}\Xi_{71})\\-\frac{1}{\sqrt{12}}(\Xi_{49}-\mathrm{i}\Xi_{98})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt6}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac12(\Xi_{20}-\mathrm{i}\Xi_{69})\\+\frac{1}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{24}-\mathrm{i}\Xi_{73})\\+\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{10}-\mathrm{i}\Xi_{59}) & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_7-\mathrm{i}\Xi_{56})\\-\frac{1}{\sqrt2}(\Xi_8-\mathrm{i}\Xi_{57})\\+\frac{1}{\sqrt6}(\Xi_9-\mathrm{i}\Xi_{58})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{12}-\mathrm{i}\Xi_{61}) & \begin{bmatrix}\frac12(\Xi_{33}+\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}+\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}-\frac12(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{38}-\mathrm{i}\Xi_{39})\\-\frac{\mathrm{i}}{2}(\Xi_{87}-\mathrm{i}\Xi_{88})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{45}+\mathrm{i}\Xi_{94}) \\
\begin{bmatrix}\frac12(\Xi_{23}-\mathrm{i}\Xi_{72})\\+\frac{1}{\sqrt{12}}(\Xi_{48}-\mathrm{i}\Xi_{97})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{24}-\mathrm{i}\Xi_{73})\\-\frac{1}{\sqrt{12}}(\Xi_{47}-\mathrm{i}\Xi_{96})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt6}(\Xi_{19}-\mathrm{i}\Xi_{68})\\-\frac{2}{\sqrt{12}}(\Xi_{21}-\mathrm{i}\Xi_{70})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{11}-\mathrm{i}\Xi_{60}) & \frac{1}{\sqrt2}(\Xi_{12}-\mathrm{i}\Xi_{61}) & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_7-\mathrm{i}\Xi_{56})\\-\frac{2}{\sqrt6}(\Xi_9-\mathrm{i}\Xi_{58})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{36}+\mathrm{i}\Xi_{37})\\-\frac{\mathrm{i}}{2}(\Xi_{85}+\mathrm{i}\Xi_{86})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{38}+\mathrm{i}\Xi_{39})\\-\frac{\mathrm{i}}{2}(\Xi_{87}+\mathrm{i}\Xi_{88})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89}) & \frac{1}{\sqrt2}(\Xi_{46}+\mathrm{i}\Xi_{95}) \\
\hline
\begin{bmatrix}\frac12(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{25}+\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}+\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{28}+\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}+\mathrm{i}\Xi_{78})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{33}+\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}+\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{36}+\mathrm{i}\Xi_{37})\\-\frac12(\Xi_{85}+\mathrm{i}\Xi_{86})\end{bmatrix} & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_{13}+\mathrm{i}\Xi_{62})\\+\frac{1}{\sqrt2}(\Xi_{14}+\mathrm{i}\Xi_{63})\\+\frac{1}{\sqrt6}(\Xi_{15}+\mathrm{i}\Xi_{64})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{16}+\mathrm{i}\Xi_{65}) & \frac{1}{\sqrt2}(\Xi_{17}+\mathrm{i}\Xi_{66}) & \frac{1}{\sqrt3}(\Xi_{47}-\mathrm{i}\Xi_{96}) \\
\begin{bmatrix}\frac12(\Xi_{25}-\mathrm{i}\Xi_{26})\\-\frac{\mathrm{i}}{2}(\Xi_{74}-\mathrm{i}\Xi_{75})\end{bmatrix} & \begin{bmatrix}-\frac12(\Xi_{27}-\mathrm{i}\Xi_{76})\\+\frac{1}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{30}+\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}+\mathrm{i}\Xi_{80})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{33}-\mathrm{i}\Xi_{34})\\-\frac{\mathrm{i}}{2}(\Xi_{82}-\mathrm{i}\Xi_{83})\end{bmatrix} & \begin{bmatrix}-\frac12(\Xi_{35}-\mathrm{i}\Xi_{84})\\+\frac{1}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{38}+\mathrm{i}\Xi_{89})\\-\frac{\mathrm{i}}{2}(\Xi_{87}+\mathrm{i}\Xi_{88})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{16}+\mathrm{i}\Xi_{65}) & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_{13}+\mathrm{i}\Xi_{62})\\-\frac{1}{\sqrt2}(\Xi_{14}+\mathrm{i}\Xi_{63})\\+\frac{1}{\sqrt6}(\Xi_{15}+\mathrm{i}\Xi_{64})\end{bmatrix} & \frac{1}{\sqrt2}(\Xi_{18}+\mathrm{i}\Xi_{67}) & \frac{1}{\sqrt3}(\Xi_{48}-\mathrm{i}\Xi_{97}) \\
\begin{bmatrix}\frac12(\Xi_{28}-\mathrm{i}\Xi_{29})\\-\frac{\mathrm{i}}{2}(\Xi_{77}-\mathrm{i}\Xi_{78})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{30}-\mathrm{i}\Xi_{31})\\-\frac{\mathrm{i}}{2}(\Xi_{79}-\mathrm{i}\Xi_{80})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{32}-\mathrm{i}\Xi_{81}) & \begin{bmatrix}\frac12(\Xi_{36}-\mathrm{i}\Xi_{37})\\-\frac{\mathrm{i}}{2}(\Xi_{85}-\mathrm{i}\Xi_{86})\end{bmatrix} & \begin{bmatrix}\frac12(\Xi_{38}-\mathrm{i}\Xi_{39})\\-\frac{\mathrm{i}}{2}(\Xi_{87}-\mathrm{i}\Xi_{88})\end{bmatrix} & -\frac{2}{\sqrt{12}}(\Xi_{40}-\mathrm{i}\Xi_{89}) & \frac{1}{\sqrt2}(\Xi_{17}+\mathrm{i}\Xi_{66}) & \frac{1}{\sqrt2}(\Xi_{18}+\mathrm{i}\Xi_{67}) & \begin{bmatrix}\frac{1}{\sqrt3}(\Xi_{13}+\mathrm{i}\Xi_{62})\\-\frac{2}{\sqrt6}(\Xi_{15}+\mathrm{i}\Xi_{64})\end{bmatrix} & \frac{1}{\sqrt3}(\Xi_{49}-\mathrm{i}\Xi_{98}) \\
\hline
\frac{1}{\sqrt2}(\Xi_{41}+\mathrm{i}\Xi_{90}) & \frac{1}{\sqrt2}(\Xi_{42}+\mathrm{i}\Xi_{91}) & \frac{1}{\sqrt2}(\Xi_{43}+\mathrm{i}\Xi_{92}) & \frac{1}{\sqrt2}(\Xi_{44}+\mathrm{i}\Xi_{93}) & \frac{1}{\sqrt2}(\Xi_{45}+\mathrm{i}\Xi_{94}) & \frac{1}{\sqrt2}(\Xi_{46}+\mathrm{i}\Xi_{95}) & \frac{1}{\sqrt3}(\Xi_{47}-\mathrm{i}\Xi_{96}) & \frac{1}{\sqrt3}(\Xi_{48}-\mathrm{i}\Xi_{97}) & \frac{1}{\sqrt3}(\Xi_{49}-\mathrm{i}\Xi_{98}) & -\Xi_0
\end{array}\right)$$

$$
\boldsymbol{\Upsilon} = \tag{4.18}
$$

$$
\left(\begin{array}{ccc|c|c}
\begin{bmatrix} \frac{1}{\sqrt{6}}(\varUpsilon_0{+}\mathrm{i}\varUpsilon_{45}) \\ +\varUpsilon_3{+}\mathrm{i}\varUpsilon_{48} \\ +\frac{1}{\sqrt{3}}(\varUpsilon_8{+}\mathrm{i}\varUpsilon_{53}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_1{-}\mathrm{i}\varUpsilon_2) \\ +\mathrm{i}(\varUpsilon_{46}{-}\mathrm{i}\varUpsilon_{47}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_4{-}\mathrm{i}\varUpsilon_5) \\ +\mathrm{i}(\varUpsilon_{49}{-}\mathrm{i}\varUpsilon_{50}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\varUpsilon_9{+}\mathrm{i}\varUpsilon_{54}) \\ +\varUpsilon_{12}{+}\mathrm{i}\varUpsilon_{57} \end{bmatrix} &
\sqrt{2}(\varUpsilon_{15}{+}\mathrm{i}\varUpsilon_{60}) \\[1ex]
\begin{bmatrix} (\varUpsilon_1{+}\mathrm{i}\varUpsilon_2) \\ +\mathrm{i}(\varUpsilon_{46}{+}\mathrm{i}\varUpsilon_{47}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\varUpsilon_0{+}\mathrm{i}\varUpsilon_{45}) \\ -\varUpsilon_3{-}\mathrm{i}\varUpsilon_{48} \\ +\frac{1}{\sqrt{3}}(\varUpsilon_8{+}\mathrm{i}\varUpsilon_{53}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_6{-}\mathrm{i}\varUpsilon_7) \\ +\mathrm{i}(\varUpsilon_{51}{-}\mathrm{i}\varUpsilon_{52}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\varUpsilon_{10}{+}\mathrm{i}\varUpsilon_{55}) \\ +\varUpsilon_{13}{+}\mathrm{i}\varUpsilon_{58} \end{bmatrix} &
\sqrt{2}(\varUpsilon_{16}{+}\mathrm{i}\varUpsilon_{61}) \\[1ex]
\begin{bmatrix} (\varUpsilon_4{+}\mathrm{i}\varUpsilon_5) \\ +\mathrm{i}(\varUpsilon_{49}{+}\mathrm{i}\varUpsilon_{50}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_6{+}\mathrm{i}\varUpsilon_7) \\ +\mathrm{i}(\varUpsilon_{51}{+}\mathrm{i}\varUpsilon_{52}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\varUpsilon_0{+}\mathrm{i}\varUpsilon_{45}) \\ -\frac{2}{\sqrt{3}}(\varUpsilon_8{+}\mathrm{i}\varUpsilon_{53}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\varUpsilon_{11}{+}\mathrm{i}\varUpsilon_{56}) \\ +\varUpsilon_{14}{+}\mathrm{i}\varUpsilon_{59} \end{bmatrix} &
\sqrt{2}(\varUpsilon_{17}{+}\mathrm{i}\varUpsilon_{62}) \\[1ex]
\hline
\begin{bmatrix} \frac{1}{\sqrt{6}}(\varUpsilon_{18}{+}\mathrm{i}\varUpsilon_{63}) \\ +\varUpsilon_{21}{+}\mathrm{i}\varUpsilon_{66} \\ +\frac{1}{\sqrt{3}}(\varUpsilon_{26}{+}\mathrm{i}\varUpsilon_{71}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_{19}{-}\mathrm{i}\varUpsilon_{20}) \\ +\mathrm{i}(\varUpsilon_{64}{-}\mathrm{i}\varUpsilon_{65}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_{22}{-}\mathrm{i}\varUpsilon_{23}) \\ +\mathrm{i}(\varUpsilon_{67}{-}\mathrm{i}\varUpsilon_{68}) \end{bmatrix} &
\sqrt{2}(\varUpsilon_{27}{+}\mathrm{i}\varUpsilon_{72}) &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\varUpsilon_9{+}\mathrm{i}\varUpsilon_{54}) \\ -\varUpsilon_{12}{-}\mathrm{i}\varUpsilon_{57} \end{bmatrix} \\[1ex]
\begin{bmatrix} (\varUpsilon_{19}{+}\mathrm{i}\varUpsilon_{20}) \\ +\mathrm{i}(\varUpsilon_{64}{+}\mathrm{i}\varUpsilon_{65}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\varUpsilon_{18}{+}\mathrm{i}\varUpsilon_{63}) \\ -\varUpsilon_{21}{-}\mathrm{i}\varUpsilon_{66} \\ +\frac{1}{\sqrt{3}}(\varUpsilon_{26}{+}\mathrm{i}\varUpsilon_{71}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_{24}{-}\mathrm{i}\varUpsilon_{25}) \\ +\mathrm{i}(\varUpsilon_{69}{-}\mathrm{i}\varUpsilon_{70}) \end{bmatrix} &
\sqrt{2}(\varUpsilon_{28}{+}\mathrm{i}\varUpsilon_{73}) &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\varUpsilon_{10}{+}\mathrm{i}\varUpsilon_{55}) \\ -\varUpsilon_{13}{-}\mathrm{i}\varUpsilon_{58} \end{bmatrix} \\[1ex]
\begin{bmatrix} (\varUpsilon_{22}{+}\mathrm{i}\varUpsilon_{23}) \\ +\mathrm{i}(\varUpsilon_{67}{+}\mathrm{i}\varUpsilon_{68}) \end{bmatrix} &
\begin{bmatrix} (\varUpsilon_{24}{+}\mathrm{i}\varUpsilon_{25}) \\ +\mathrm{i}(\varUpsilon_{69}{+}\mathrm{i}\varUpsilon_{70}) \end{bmatrix} &
\begin{bmatrix} \frac{1}{\sqrt{6}}(\varUpsilon_{18}{+}\mathrm{i}\varUpsilon_{63}) \\ -\frac{2}{\sqrt{3}}(\varUpsilon_{26}{+}\mathrm{i}\varUpsilon_{71}) \end{bmatrix} &
\sqrt{2}(\varUpsilon_{29}{+}\mathrm{i}\varUpsilon_{74}) &
\begin{bmatrix} \frac{1}{\sqrt{2}}(\varUpsilon_{11}{+}\mathrm{i}\varUpsilon_{56}) \\ -\varUpsilon_{14}{-}\mathrm{i}\varUpsilon_{59} \end{bmatrix} \\[1ex]
\hline
\begin{bmatrix} \sqrt{\frac{2}{3}}(\varUpsilon_{30}{+}\mathrm{i}\varUpsilon_{75}) \\ +\varUpsilon_{31}{+}\mathrm{i}\varUpsilon_{76} \\ +\frac{1}{\sqrt{3}}(\varUpsilon_{32}{+}\mathrm{i}\varUpsilon_{77}) \end{bmatrix} &
\begin{bmatrix} \varUpsilon_{33}{+}\mathrm{i}\varUpsilon_{78} \\ -\frac{1}{\sqrt{2}}(\varUpsilon_{11}{+}\mathrm{i}\varUpsilon_{56}) \end{bmatrix} &
\begin{bmatrix} \varUpsilon_{34}{+}\mathrm{i}\varUpsilon_{79} \\ +\frac{1}{\sqrt{2}}(\varUpsilon_{10}{+}\mathrm{i}\varUpsilon_{55}) \end{bmatrix} &
\sqrt{2}(\varUpsilon_{36}{+}\mathrm{i}\varUpsilon_{81}) &
\sqrt{2}(\varUpsilon_{39}{+}\mathrm{i}\varUpsilon_{84}) \\[1ex]
\begin{bmatrix} \varUpsilon_{33}{+}\mathrm{i}\varUpsilon_{78} \\ +\frac{1}{\sqrt{2}}(\varUpsilon_{11}{+}\mathrm{i}\varUpsilon_{56}) \end{bmatrix} &
\begin{bmatrix} \sqrt{\frac{2}{3}}(\varUpsilon_{30}{+}\mathrm{i}\varUpsilon_{75}) \\ -\varUpsilon_{31}{-}\mathrm{i}\varUpsilon_{76} \\ +\frac{1}{\sqrt{3}}(\varUpsilon_{32}{+}\mathrm{i}\varUpsilon_{77}) \end{bmatrix} &
\begin{bmatrix} \varUpsilon_{35}{+}\mathrm{i}\varUpsilon_{80} \\ -\frac{1}{\sqrt{2}}(\varUpsilon_9{+}\mathrm{i}\varUpsilon_{54}) \end{bmatrix} &
\sqrt{2}(\varUpsilon_{37}{+}\mathrm{i}\varUpsilon_{82}) &
\sqrt{2}(\varUpsilon_{40}{+}\mathrm{i}\varUpsilon_{85}) \\[1ex]
\begin{bmatrix} \varUpsilon_{34}{+}\mathrm{i}\varUpsilon_{79} \\ -\frac{1}{\sqrt{2}}(\varUpsilon_{10}{+}\mathrm{i}\varUpsilon_{55}) \end{bmatrix} &
\begin{bmatrix} \varUpsilon_{35}{+}\mathrm{i}\varUpsilon_{80} \\ +\frac{1}{\sqrt{2}}(\varUpsilon_9{+}\mathrm{i}\varUpsilon_{54}) \end{bmatrix} &
\begin{bmatrix} \sqrt{\frac{2}{3}}(\varUpsilon_{30}{+}\mathrm{i}\varUpsilon_{75}) \\ -\frac{2}{\sqrt{3}}(\varUpsilon_{32}{+}\mathrm{i}\varUpsilon_{77}) \end{bmatrix} &
\sqrt{2}(\varUpsilon_{38}{+}\mathrm{i}\varUpsilon_{83}) &
\sqrt{2}(\varUpsilon_{41}{+}\mathrm{i}\varUpsilon_{88}) \\[1ex]
\hline
\sqrt{2}(\varUpsilon_{42}{-}\mathrm{i}\varUpsilon_{87}) &
\sqrt{2}(\varUpsilon_{43}{-}\mathrm{i}\varUpsilon_{88}) &
\sqrt{2}(\varUpsilon_{44}{-}\mathrm{i}\varUpsilon_{89}) &
-\frac{3}{\sqrt{6}}(\varUpsilon_{18}{+}\mathrm{i}\varUpsilon_{63}) &
\frac{3}{\sqrt{6}}(\varUpsilon_0{+}\mathrm{i}\varUpsilon_{45})
\end{array}\right)
$$

$$
= \begin{pmatrix}
\boldsymbol{\Upsilon}_A & \boldsymbol{\Upsilon}_a + \boldsymbol{\Upsilon}_b & \boldsymbol{\Upsilon}_c \\
\boldsymbol{\Upsilon}_B & \boldsymbol{\Upsilon}_d & \boldsymbol{\Upsilon}_a - \boldsymbol{\Upsilon}_b \\
\boldsymbol{\Upsilon}_C - \varepsilon(\boldsymbol{\Upsilon}_a) & \boldsymbol{\Upsilon}_e & \boldsymbol{\Upsilon}_f \\
\boldsymbol{\Upsilon}_g^* & -\operatorname{tr}(\boldsymbol{\Upsilon}_B) & \operatorname{tr}(\boldsymbol{\Upsilon}_A)
\end{pmatrix} ,
$$

$(\varepsilon(A))_{\alpha\beta} = \sum_{\gamma=1}^3 \varepsilon_{\alpha\beta\gamma} A_\gamma$.

Then, using

$$\begin{aligned} &\mathrm{tr}(\sigma^a\sigma^b) = 2\,\delta^{ab}\;, \qquad \mathrm{tr}(\lambda^a\lambda^b) = 2\,\delta^{ab}\;, \\ &\mathrm{tr}_c((\gamma^\kappa\wedge\gamma^\lambda)(\gamma^\mu\wedge\gamma^\nu)) = 4(\delta^{\lambda\mu}\delta^{\kappa\nu}-\delta^{\kappa\mu}\delta^{\lambda\nu})\;,\; \mathrm{tr}_c(\gamma^\mu\gamma^\nu) = 4\delta^{\mu\nu}\;,\; \mathrm{tr}_c(1) = 4\;, \end{aligned} \tag{4.22}$$

we obtain for (4.10b)

$$\begin{aligned} \mathcal{L}_2 = \tfrac{1}{4}\delta^{\kappa\mu}\delta^{\lambda\nu}\big(&\textstyle\sum_{a=1}^8 G^a_{\kappa\lambda}G^a_{\mu\nu} + \sum_{a=1}^3 W^a_{\kappa\lambda}W^a_{\mu\nu} + A'_{\kappa\lambda}A'_{\mu\nu} + \tilde{A}_{\kappa\lambda}\tilde{A}_{\mu\nu} \\ &\textstyle+ \sum_{a=1}^6 \partial_{[\kappa}X^a_{\lambda]}\,\partial_{[\mu}X^a_{\nu]} + \sum_{a=1}^6 \partial_{[\kappa}Y^a_{\lambda]}\,\partial_{[\mu}Y^a_{\nu]}\big) + I.T\;. \end{aligned} \tag{4.23}$$

We proceed with the calculation of $\mathcal{L}_1$, where we restrict ourselves again to the interesting part. Using (4.16a) and (4.19a) we get

$$\begin{aligned} &\mathbf{d\Psi} + [A,\mathbf{\Psi}+\mathbf{m}] = \mathbf{d\Psi} + [A,\mathbf{m}] + I.T = \\ &\begin{pmatrix} -\sqrt{\frac{4}{15}}\partial_\mu\Psi_0\mathbb{1}_3 + \sum_{a=1}^8 \partial_\mu\mathbf{\Psi}_a\,\lambda^a & \mathrm{i}\frac{g_0}{2}\mathbf{X}_\mu \\ (\mathrm{i}\frac{g_0}{2}\mathbf{X}_\mu)^* & \sqrt{\frac{3}{5}}\partial_\mu\Psi_0\mathbb{1}_2 + \sum_{a=1}^3 \partial_\mu\mathbf{\Psi}'_a\,\sigma^a \end{pmatrix}\gamma^\mu + I.T\,. \end{aligned} \tag{4.24}$$

Now, using (4.19a) and (4.16d) we calculate

$$\mathbf{d\Phi} + (A{+}A''\mathbb{1}_5)(\mathbf{\Phi}{+}\mathbf{n}) = \mathbf{d\Phi} + (A{+}A''\mathbb{1}_5)\mathbf{n} + I.T = \begin{pmatrix} D_\mu\mathbf{\Phi}_g \\ D_\mu\mathbf{\Phi}_w \end{pmatrix}\gamma^\mu\;, \tag{4.25}$$

$$\begin{aligned} D_\mu\mathbf{\Phi}_g &= \partial_\mu\mathbf{\Phi}_g + \mathrm{i}\tfrac{g_0}{2}X_\mu + I.T\;, \\ D_\mu\mathbf{\Phi}_w &= \begin{pmatrix} \partial_\mu\Phi_0 + \mathrm{i}\frac{g_0}{2}(W^3_\mu - (\sqrt{\frac{3}{5}}A'_\mu - \sqrt{\frac{2}{5}}\tilde{A}_\mu)) + I.T \\ \mathrm{i}\frac{g_0}{2}(W^1_\mu + \mathrm{i}W^2_\mu) + I.T \end{pmatrix}\;. \end{aligned}$$

Next, using (4.19a) and (4.18) we calculate

$$\begin{aligned} &\mathbf{d\Upsilon} + \check{A}(\mathbf{\Upsilon}{+}\mathbf{n}') - (\mathbf{\Upsilon}{+}\mathbf{n}')A + A''(\mathbf{\Upsilon}{+}\mathbf{n}') = \mathbf{d\Upsilon} + \check{A}\mathbf{n}'{-}\mathbf{n}'A{+}A''\mathbf{n}' + I.T \\ &= \begin{pmatrix} \begin{bmatrix} \partial_\mu\mathbf{\Upsilon}_A + \mathrm{i}\frac{g_0}{2}(\sqrt{\frac{2}{5}}\tilde{A}_\mu \\ -\frac{3}{\sqrt{15}}A'_\mu + W^3_\mu)\mathbb{1}_3 \end{bmatrix} & \begin{bmatrix} \partial_\mu\mathbf{\Upsilon}_a + \mathrm{i}\frac{g_0}{2}X_\mu + \\ \partial_\mu\mathbf{\Upsilon}_b + \mathrm{i}\frac{g_0}{2}(-2X_\mu) \end{bmatrix} & \partial_\mu\mathbf{\Upsilon}_c + \mathrm{i}\frac{g_0}{2}(-4Y_\mu) \\ \partial_\mu\mathbf{\Upsilon}_B + \mathrm{i}\frac{g_0}{2}(W^1_\mu + \mathrm{i}W^2_\mu)\mathbb{1}_3 & \partial_\mu\mathbf{\Upsilon}_d & \begin{bmatrix} \partial_\mu\mathbf{\Upsilon}_a + \mathrm{i}\frac{g_0}{2}X_\mu \\ -\partial_\mu\mathbf{\Upsilon}_b - \mathrm{i}\frac{g_0}{2}(-2X_\mu) \end{bmatrix} \\ \partial_\mu\mathbf{\Upsilon}_C - \varepsilon(\partial_\mu\mathbf{\Upsilon}_a + \mathrm{i}\frac{g_0}{2}X_\mu) & \partial_\mu\mathbf{\Upsilon}_e & \partial_\mu\mathbf{\Upsilon}_f \\ (\partial_\mu\mathbf{\Upsilon}_g + \mathrm{i}\frac{g_0}{2}(4Y_\mu))^* & \begin{bmatrix} -\mathrm{tr}(\partial_\mu\mathbf{\Upsilon}_B) - \\ 3\mathrm{i}\frac{g_0}{2}(W^1_\mu + \mathrm{i}W^2_\mu) \end{bmatrix} & \begin{bmatrix} \mathrm{tr}(\partial_\mu\mathbf{\Upsilon}_A) + 3\mathrm{i}\frac{g_0}{2}(-\frac{3}{\sqrt{15}}A'_\mu \\ +\sqrt{\frac{2}{5}}\tilde{A}_\mu + W^3_\mu) \end{bmatrix} \end{pmatrix}\gamma^\mu + I.T. \end{aligned} \tag{4.26}$$

Finally, using (4.19a) and (4.16e) we calculate

$$\begin{aligned} &\mathbf{d\Xi} + \check{A}(\mathbf{\Xi}{+}\mathbf{m}') + (\mathbf{\Xi}{+}\mathbf{m}')\check{A}^T - A''(\mathbf{\Xi}{+}\mathbf{m}') = \mathbf{d\Xi}{+}\check{A}\mathbf{m}'{+}\mathbf{m}'\check{A}^T{-}A''\mathbf{m}'{+}I.T \\ &= \begin{pmatrix} \overline{\partial_\mu\mathbf{\Xi}_A} & \overline{\partial_\mu\mathbf{\Xi}_D} - \frac{1}{2}\varepsilon(\partial_\mu\overline{\mathbf{\Xi}_c}) & (\partial_\mu\mathbf{\Xi}^0_E)^* & \partial_\mu\mathbf{\Xi}_a + \mathrm{i}\frac{g_0}{2}Y_\mu \\ \overline{\partial_\mu\mathbf{\Xi}_D} + \frac{1}{2}\varepsilon(\partial_\mu\overline{\mathbf{\Xi}_c}) & \overline{\partial_\mu\mathbf{\Xi}_B} & (\partial_\mu\mathbf{\Xi}^0_F)^* & \partial_\mu\mathbf{\Xi}_b - \mathrm{i}\frac{g_0}{2}X_\mu \\ \overline{\partial_\mu\mathbf{\Xi}^0_E} & \overline{\mathbf{\Xi}^0_F} & \partial_\mu\mathbf{\Xi}_C & \overline{\partial_\mu\mathbf{\Xi}_c} \\ (\partial_\mu\mathbf{\Xi}_a + \mathrm{i}\frac{g_0}{2}Y_\mu)^T & (\partial_\mu\mathbf{\Xi}_b - \mathrm{i}\frac{g_0}{2}X_\mu)^T & (\partial_\mu\mathbf{\Xi}_c)^* & -(\partial_\mu\mathbf{\Xi}_0 - \mathrm{i}\frac{g_0}{2}(\frac{12}{\sqrt{15}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)) \end{pmatrix}\gamma^\mu \\ &+ I.T\;. \end{aligned} \tag{4.27}$$

The Lagrangian $\mathcal{L}_1$ is obtained from formulae (4.24), (4.25), (4.26) and (4.27), where one has to use (4.22):

$$\begin{aligned}\mathcal{L}_1 &= \tfrac{8\mu_0}{g_0^2}\delta^{\mu\nu}\big(\textstyle\sum_{a=0}^8 \partial_\mu\boldsymbol{\Psi}_a\,\partial_\nu\boldsymbol{\Psi}_a + \sum_{a=1}^3 \partial_\mu\boldsymbol{\Psi}'_a\,\partial_\nu\boldsymbol{\Psi}'_a\big) + \tfrac{4\mu_1}{g_0^2}\delta^{\mu\nu}\ \textstyle\sum_{a=0}^6 \partial_\mu\varPhi_a\,\partial_\nu\varPhi_a \\ &\quad + \tfrac{8\mu_2}{g_0^2}\delta^{\mu\nu}\ \textstyle\sum_{i=0}^{89}\partial_\mu\varUpsilon_i\,\partial_\nu\varUpsilon_i + \tfrac{4\mu_3}{g_0^2}\delta^{\mu\nu}\ \sum_{i=0}^{98}\partial_\mu\varXi_i\,\partial_\nu\varXi_i \\ &\quad + (\mu_1+12\mu_2)\delta^{\mu\nu}\big(W^1_\mu W^1_\nu + W^2_\mu W^2_\nu + \\ &\qquad\qquad + (W^3_\mu - \sqrt{\tfrac{3}{5}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)(W^3_\nu - \sqrt{\tfrac{3}{5}}A'_\nu + \sqrt{\tfrac{2}{5}}\tilde{A}_\nu)\big) \\ &\quad + \mu_3\delta^{\mu\nu}(4\sqrt{\tfrac{3}{5}}A'_\mu + \sqrt{\tfrac{2}{5}}\tilde{A}_\mu)(4\sqrt{\tfrac{3}{5}}A'_\nu + \sqrt{\tfrac{2}{5}}\tilde{A}_\nu) \\ &\quad + \delta^{\mu\nu}\textstyle\sum_{a=1}^6\big((2\mu_0{+}\mu_1{+}12\mu_2{+}2\mu_3)X^a_\mu X^a_\nu + (2\mu_0{+}32\mu_2{+}2\mu_3)Y^a_\mu Y^a_\nu\big) + I.T\ .\end{aligned} \tag{4.28}$$

We perform the orthogonal transformation by Euler angles

$$\begin{pmatrix}Z_\mu\\ Z'_\mu\\ P_\mu\end{pmatrix} = \begin{pmatrix}\cos\phi_E & -\sin\phi_E & 0\\ \sin\phi_E & \cos\phi_E & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta_E & -\sin\theta_E\\ 0 & \sin\theta_E & \cos\theta_E\end{pmatrix}\begin{pmatrix}\cos\psi_E & -\sin\psi_E & 0\\ \sin\psi_E & \cos\psi_E & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}W^3_\mu\\ A'_\mu\\ \tilde{A}_\mu\end{pmatrix}. \tag{4.29a}$$

The photon P_μ is the massless linear combination, which is perpendicular to the plane spanned by $(W^3_\mu - \sqrt{\frac{3}{5}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)$ and $(4\sqrt{\frac{3}{5}}A'_\mu + \sqrt{\frac{2}{5}}\tilde{A}_\mu)$, see (4.28). Calculating the vector product yields immediately

$$P_\mu = \sqrt{\tfrac{3}{8}}W^3_\mu + \sqrt{\tfrac{1}{40}}A'_\mu - \sqrt{\tfrac{3}{5}}\tilde{A}_\mu\ , \tag{4.29b}$$

which implies

$$\cos\theta_E = -\sqrt{\tfrac{3}{5}}\ ,\quad \sin\theta_E = \sqrt{\tfrac{2}{5}}\ ,\qquad \cos\psi_E = \tfrac{1}{4}\ ,\quad \sin\psi_E = \sqrt{\tfrac{15}{16}}\ . \tag{4.29c}$$

The Euler angle ϕ_E is determined by the diagonalization of the mass matrix. The result is

$$\tan 2\phi_E = -\tfrac{3}{4} + \tfrac{25}{4}\lambda_4\ ,\qquad \lambda_4 := \tfrac{(\mu_1+12\mu_2)}{25\mu_3}\ . \tag{4.29d}$$

We choose $\cos\phi_E < 0$ and $\sin\phi_E > 0$. Then, the inverse transformation is for $\lambda_4 \ll 1$ given by

$$\begin{aligned} W^3_\mu &= \sqrt{\tfrac{5}{8}}Z_\mu + \sqrt{\tfrac{3}{8}}P_\mu - \sqrt{\tfrac{5}{2}}\lambda_4 Z'_\mu\ ,\\ A'_\mu &= -\sqrt{\tfrac{3}{200}}(1-16\lambda_4)Z_\mu + \sqrt{\tfrac{1}{40}}P_\mu + \sqrt{\tfrac{24}{25}}(1+\tfrac{1}{4}\lambda_4)Z'_\mu\ ,\\ \tilde{A}_\mu &= \tfrac{3}{5}(1+\tfrac{2}{3}\lambda_4)Z_\mu - \sqrt{\tfrac{3}{5}}P_\mu + \tfrac{1}{5}(1-6\lambda_4)Z'_\mu\ .\end{aligned} \tag{4.29e}$$

The Lagrangian (4.28) requires to perform the reparametrizations

$$\begin{aligned} \varPsi_i &= \tfrac{g_0}{\sqrt{16\,\mu_0}}\psi_i\ , && i=0,\dots,8\ , && \varPsi'_i = \tfrac{g_0}{\sqrt{16\,\mu_0}}\psi'_i\ , && i=1,\dots,3\ ,\\ \varPhi_i &= \tfrac{g_0}{\sqrt{8\,\mu_1}}\phi_i\ , && i=0,\dots,6\ , && && \\ \varUpsilon_i &= \tfrac{g_0}{\sqrt{16\,\mu_2}}\upsilon_i\ , && i=0,\dots,89\ , && \varXi_i = \tfrac{g_0}{\sqrt{8\,\mu_3}}\xi_i\ , && i=0,\dots,98\ .\end{aligned} \tag{4.30}$$

It remains to compute the quadratic terms of the Higgs potential (4.10d). Due to the extremely rich Higgs structure we need computer algebra for that calculation.

It turns out that it is advantageous to perform an orthogonal transformation in the ϕ_0–υ_0–sector:

$$\begin{pmatrix}\phi_0\\ \upsilon_0\end{pmatrix}=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_0'\\ \upsilon_0'\end{pmatrix}, \qquad \tan\alpha=\sqrt{\tfrac{12\mu_2}{\mu_1}}\;. \tag{4.31}$$

The motivation for this transformation is that the linear combination ϕ_0' receives a much smaller mass than all other Higgs fields, see below. We present the quadratic terms of the Higgs potential in Appendix C.

We perform a Wick rotation from the Riemannian manifold X to the Minkowskian manifold X_M by introduction of a global minus sign in the action and by replacing[2]

$$\delta^{\mu\nu}\mapsto -g^{\mu\nu}\;, \qquad g^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)\;. \tag{4.32}$$

We define $P_{\mu\nu}:=\partial_{[\mu}P_{\nu]}$ and

$$\begin{aligned}
m_W^2&=(2\,\mu_1+24\,\mu_2)\;, & m_Z^2&=\tfrac{1}{\cos^2(\theta_W-\theta_W')}m_W^2\;,\\
m_{Z'}^2&=32\mu_3\cos^2(\theta_W-\theta_W')\;, & &\\
m_X^2&=(4\,\mu_0+2\,\mu_1+24\,\mu_2+4\mu_3)\;, & m_Y^2&=(4\,\mu_0+64\,\mu_2+4\mu_3)\;,\\
\sin\theta_W&=\sqrt{\tfrac{3}{8}}\;, & \theta_W'&=\tfrac{1}{2}\sqrt{\tfrac{5}{3}}\lambda_4\;.
\end{aligned}\tag{4.33}$$

Now we can write down the final formula for the bosonic Lagrangian:

$$\begin{aligned}
\mathcal{L}&=-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}(\textstyle\sum_{a=1}^{8}(G^a_{\kappa\lambda}G^a_{\mu\nu})+P_{\kappa\lambda}P_{\mu\nu})\\
&\quad+\textstyle\sum_{a=1}^{2}(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\partial_{[\kappa}W^a_{\lambda]}\,\partial_{[\mu}W^a_{\nu]}+\tfrac{1}{2}g^{\mu\nu}m_W^2W^a_\mu W^a_\nu)\\
&\quad+(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\partial_{[\kappa}Z_{\lambda]}\,\partial_{[\mu}Z_{\nu]}+\tfrac{1}{2}g^{\mu\nu}m_Z^2Z_\mu Z_\nu)\\
&\quad+(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\partial_{[\kappa}Z'_{\lambda]}\,\partial_{[\mu}Z'_{\nu]}+\tfrac{1}{2}g^{\mu\nu}m_{Z'}^2Z'_\mu Z'_\nu)+\mathcal{L}_{ew}(P,W,Z,Z')\\
&\quad+\textstyle\sum_{a=1}^{6}(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\partial_{[\kappa}X^a_{\lambda]}\,\partial_{[\mu}X^a_{\nu]}+\tfrac{1}{2}g^{\mu\nu}m_X^2X^a_\mu X^a_\nu)\\
&\quad+\textstyle\sum_{a=1}^{6}(-\tfrac{1}{4}g^{\kappa\mu}g^{\lambda\nu}\partial_{[\kappa}Y^a_{\lambda]}\,\partial_{[\mu}Y^a_{\nu]}+\tfrac{1}{2}g^{\mu\nu}m_Y^2Y^a_\mu Y^a_\nu)+\mathcal{L}_H+I.T\;,
\end{aligned}\tag{4.34a}$$

where

$$\begin{aligned}
&\mathcal{L}_{ew}(P,W,Z,Z')\\
&\quad=g_0\,g^{\kappa\mu}g^{\lambda\nu}(\partial_{[\kappa}W^1_{\lambda]}\,W^2_\mu W^3_\nu+\partial_{[\kappa}W^2_{\lambda]}\,W^3_\mu W^1_\nu+\partial_{[\kappa}W^3_{\lambda]}\,W^1_\mu W^2_\nu)\\
&\quad-\tfrac{1}{2}\,g_0^2(g^{\kappa\mu}g^{\lambda\nu}-g^{\kappa\nu}g^{\lambda\mu})(W^1_\kappa W^1_\mu W^2_\lambda W^2_\nu+W^1_\kappa W^1_\mu W^3_\lambda W^3_\nu+W^2_\kappa W^2_\mu W^3_\lambda W^3_\nu)\;,
\end{aligned}\tag{4.34b}$$

$$\begin{aligned}
\mathcal{L}_H=\tfrac{1}{2}g^{\mu\nu}\big(&\textstyle\sum_{i=0}^{8}\partial_\mu\psi_i\,\partial_\nu\psi_i+\sum_{i=1}^{3}\partial_\mu\psi_i'\,\partial_\nu\psi_i'+\partial_\mu\phi_0'\,\partial_\nu\phi_0'+\partial_\mu\upsilon_0'\,\partial_\nu\upsilon_0'\\
&+\textstyle\sum_{i=1}^{6}\partial_\mu\phi_i\,\partial_\nu\phi_i+\sum_{i=0}^{98}\partial_\mu\xi_i\,\partial_\nu\xi_i+\sum_{i=1}^{89}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i\big)-\mathcal{L}_0\;.
\end{aligned}\tag{4.34c}$$

This is precisely the bosonic Lagrangian of the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$–model, where the masses of the gauge bosons are given in (4.33). The parameters μ_1,μ_2,μ_3 and the Weinberg angle θ_W will be determined in Section 4.4 when discussing the fermionic action. Within our framework there is no possibility to determine μ_0. However, we will find in Section 4.4 that the X and Y bosons lead to proton decay. In order to suppress the proton decay sufficiently we need $\mu_0\gg\max(\mu_1,\mu_2)$. Then, it remains to derive the masses of gauge and Higgs bosons in Section 5.

[2]The minus sign in $\delta^{\mu\nu}\mapsto -g^{\mu\nu}$ is due to $(\hat\gamma^5)^*=-\hat\gamma^5$ on the Minkowski space.

4.4. **The Fermionic Action.** Now we write down the fermionic action S_F defined in (2.10). However, we pass immediately to the Minkowski space X_M. We denote the gamma matrices in Minkowski space by $\{\hat{\gamma}^\mu\}$ and use the convention

$$\hat{\gamma}^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \; \hat{\gamma}^a = \begin{pmatrix} 0 & -\sigma^a \\ \sigma^a & 0 \end{pmatrix}, \; \hat{\gamma}^5 = \mathrm{i}\hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}. \tag{4.35}$$

Then, the invariant fermionic action is

$$S_F = \tfrac{1}{4} \int_{X_M} dx\, \boldsymbol{\psi}^* \hat{\gamma}^0 (D + \mathrm{i}\rho_M) \boldsymbol{\psi}\;. \tag{4.36}$$

The factor $\frac{1}{4}$ additional to (2.10) occurs because we are going to impose constraints on $\boldsymbol{\psi}$, which require precisely the form (4.36) for the action, see below. More explicitly, inserting (4.1) and (4.2) and using (3.17) we obtain

$$D + \mathrm{i}\rho_M = \tag{4.37a}$$
$$\begin{pmatrix} \mathsf{D} + \mathrm{i}\tilde{\pi}(A{+}A'') & -\hat{\gamma}^5\tilde{\pi}(\mathbf{\Psi}+\mathbf{m}) & -\hat{\gamma}^5\tilde{\pi}(\tilde{\mathbf{\Phi}}{+}\tilde{\mathbf{\Xi}}{+}\tilde{\mathbf{\Upsilon}}) & 0 \\ \hat{\gamma}^5\tilde{\pi}(\tilde{\mathbf{\Psi}})^* & \mathsf{D} + \mathrm{i}\tilde{\pi}(A{+}A'') & 0 & -\hat{\gamma}^5\tilde{\pi}(\tilde{\mathbf{\Phi}}{+}\tilde{\mathbf{\Xi}}{+}\tilde{\mathbf{\Upsilon}}) \\ \hat{\gamma}^5\tilde{\pi}(\tilde{\mathbf{\Phi}}{+}\tilde{\mathbf{\Xi}}{+}\tilde{\mathbf{\Upsilon}})^* & 0 & \mathsf{D} - \hat{\gamma}^2\overline{(\mathrm{i}\tilde{\pi}(A{+}A''))}\hat{\gamma}^2 & -\hat{\gamma}^5\overline{\tilde{\pi}(\tilde{\mathbf{\Psi}})} \\ 0 & \hat{\gamma}^5\tilde{\pi}(\tilde{\mathbf{\Phi}}{+}\tilde{\mathbf{\Xi}}{+}\tilde{\mathbf{\Upsilon}})^* & \hat{\gamma}^5\tilde{\pi}(\tilde{\mathbf{\Psi}})^T & \mathsf{D} - \hat{\gamma}^2\overline{(\mathrm{i}\tilde{\pi}(A{+}A''))}\hat{\gamma}^2 \end{pmatrix},$$

where

$$\begin{aligned} \tilde{\pi}(A{+}A'') &:= \mathrm{diag}\big((\pi_{10}(A){-}\tfrac{1}{2}A''\mathbb{1}_{10})\otimes\mathbb{1}_3\,,\; \hat{\gamma}^2\overline{(\pi_5(A){-}\tfrac{3}{2}A''\mathbb{1}_5)}\hat{\gamma}^2\otimes\mathbb{1}_3\,,\; -\tfrac{5}{2}A''\otimes\mathbb{1}_3\big)\,, \\ \tilde{\pi}(\tilde{\mathbf{\Psi}}) &:= \mathrm{diag}\,\big((\check{\mathbf{\Psi}}+\check{\mathbf{m}})\otimes M_{10}\;,\; -\overline{(\mathbf{\Psi}+\mathbf{m})\otimes M_5}\;,\; 0_{3\times 3}\big)\;, \end{aligned} \tag{4.37b}$$

$$\tilde{\pi}(\tilde{\mathbf{\Phi}}{+}\tilde{\mathbf{\Xi}}{+}\tilde{\mathbf{\Upsilon}}) := \begin{pmatrix} \begin{bmatrix} (\hat{\mathbf{\Phi}}+\hat{\mathbf{n}})\otimes M_d \\ +(\mathbf{\Xi}+\mathbf{m}')\otimes M_N \end{bmatrix} & \begin{bmatrix} (\check{\mathbf{\Phi}}+\check{\mathbf{n}})\otimes M_{\tilde{u}} \\ +(\mathbf{\Upsilon}+\mathbf{n}')\otimes M_{\tilde{n}} \end{bmatrix} & 0 \\ \begin{bmatrix} (\check{\mathbf{\Phi}}+\check{\mathbf{n}})^T\otimes M_{\tilde{u}}^T \\ +(\mathbf{\Upsilon}+\mathbf{n}')^T\otimes M_{\tilde{n}}^T \end{bmatrix} & 0 & \overline{(\mathbf{\Phi}+\mathbf{n})}\otimes M_e \\ 0 & (\mathbf{\Phi}+\mathbf{n})^*\otimes M_e^T & 0 \end{pmatrix}.$$

We have used that within our convention (4.35) we have $\hat{\gamma}^5 = -(\hat{\gamma}^5)^*$ and $[\mathsf{D},\bar{f}] = -\hat{\gamma}^2\overline{[\mathsf{D},f]}\hat{\gamma}^2$. Recall [13] that $\tilde{\pi}(A{+}A'')$ is given by commutators of $\mathsf{D}\otimes\mathbb{1}_{192}$ with an arbitrary number of elements of the form $f\otimes\hat{\pi}(a)$, where $a\in\mathfrak{a}$ and $f\in C^\infty(X)$. This fact and the complex conjugation in (3.3) are the reasons why terms of the form $[\mathsf{D},\bar{f}]$ occur in $\tilde{\pi}(A{+}A'')$.

Minkowskian fermions $\boldsymbol{\psi}$ live in the space $h_M = L^2(X_M,S)\otimes\mathbb{C}^{192}$ and have in terms of the decomposition (4.37a) the form

$$\boldsymbol{\psi} = (\boldsymbol{\psi}_1,\boldsymbol{\psi}_2,\boldsymbol{\psi}_3,\boldsymbol{\psi}_4)^T\;,\quad \boldsymbol{\psi}_i \in L^2(X_M,S)\otimes\mathbb{C}^{48}\;. \tag{4.38}$$

However, we shall restrict ourselves to the subspace of h_M invariant under the charge conjugation $\mathcal{C}$, the chirality operator $\tilde{\Gamma}$ and a symmetry transformation

$\mathcal{S}$ defined in terms of 48×48–blocks by

$$\mathcal{C} := \begin{pmatrix} 0 & 0 & -\hat{\gamma}^2\otimes \mathbb{1}_{48} & 0 \\ 0 & 0 & 0 & -\hat{\gamma}^2\otimes \mathbb{1}_{48} \\ -\hat{\gamma}^2\otimes \mathbb{1}_{48} & 0 & 0 & 0 \\ 0 & -\hat{\gamma}^2\otimes \mathbb{1}_{48} & 0 & 0 \end{pmatrix} \circ \text{ c.c} \;,\; \mathcal{S} := \begin{pmatrix} 0 & \mathbb{1}_{48} & 0 & 0 \\ \mathbb{1}_{48} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{48} \\ 0 & 0 & \mathbb{1}_{48} & 0 \end{pmatrix} ,$$

$$\tilde{\Gamma} := \mathrm{diag}(-\hat{\gamma}^5\otimes \mathbb{1}_{48} \,,\; -\hat{\gamma}^5\otimes \mathbb{1}_{48} \,,\; \hat{\gamma}^5\otimes \mathbb{1}_{48} \,,\; \hat{\gamma}^5\otimes \mathbb{1}_{48}) \;, \tag{4.39}$$

where c.c means complex conjugation. Thus, we consider elements $\boldsymbol{\psi}\in h_M$ of the form

$$\boldsymbol{\psi} = \mathcal{C}\boldsymbol{\psi} = \hat{\Gamma}\boldsymbol{\psi} = \mathcal{S}\boldsymbol{\psi} = \begin{pmatrix} \frac{1}{2}(1-\hat{\gamma}^5)\boldsymbol{\psi}_1 \\ \frac{1}{2}(1-\hat{\gamma}^5)\boldsymbol{\psi}_1 \\ -\frac{1}{2}(1+\hat{\gamma}^5)\hat{\gamma}^2\bar{\boldsymbol{\psi}}_1 \\ -\frac{1}{2}(1+\hat{\gamma}^5)\hat{\gamma}^2\bar{\boldsymbol{\psi}}_1 \end{pmatrix} \equiv \begin{pmatrix} \frac{1}{2}(1-\hat{\gamma}^5)\boldsymbol{\psi}_1 \\ \frac{1}{2}(1-\hat{\gamma}^5)\boldsymbol{\psi}_1 \\ -\hat{\gamma}^2\overline{\frac{1}{2}(1-\hat{\gamma}^5)\boldsymbol{\psi}_1} \\ -\hat{\gamma}^2\overline{\frac{1}{2}(1-\hat{\gamma}^5)\boldsymbol{\psi}_1} \end{pmatrix} . \tag{4.40}$$

Observe that the choice (4.39) for the chirality operator breaks the structure of the model, which is precisely our intention. Since $\tilde{\Gamma}$ commutes with $\hat{\pi}(\mathfrak{a})$, the gauge invariance is not destroyed. But $\tilde{\Gamma}$ no longer anticommutes with the whole D. We see that D – applied on chiral fermions (4.40) –

$$\tfrac{1}{2}(\mathrm{id}_h - \tilde{\Gamma})D\tfrac{1}{2}(\mathrm{id}_h + \tilde{\Gamma}) \;,$$

differs from the matrix (4.37a) by the absence of $\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Psi}})$. In other words, the choice (4.39) for the chirality condition eliminates the disturbing terms $\hat{\gamma}^5\tilde{\pi}(\tilde{\boldsymbol{\Psi}})$ in the fermionic action.

Within our conventions one has the block structure

$$\tfrac{1}{2}(1-\hat{\gamma}^5)\boldsymbol{\psi}_1 = \begin{pmatrix} 0 \\ \boldsymbol{\psi}_0 \end{pmatrix} , \quad \boldsymbol{\psi}_0 \in L^2(X_M)\otimes\mathbb{C}^2\otimes\mathbb{C}^{48} \;, \tag{4.41}$$

where $L^2(X_M)$ denotes the space of square integrable functions on the Minkowski space. In local bases we have

$$\mathsf{D} = \mathrm{i}\hat{\gamma}^\mu\partial_\mu \;, \quad A = A_\mu\hat{\gamma}^\mu \;, \quad A'' = A''_\mu\hat{\gamma}^\mu \;. \tag{4.42}$$

We define $\sigma^0 = \tilde{\sigma}^0 = \mathbb{1}_2$ and $\tilde{\sigma}^a = -\sigma^a$, $a = 1,2,3$, or in a symbolic notation

$$\sigma^\mu = \{\mathbb{1}_2, \sigma^a\} \;, \quad \tilde{\sigma}^\mu = \{\mathbb{1}_2, -\sigma^a\} \;, \qquad \mu = 0,1,2,3 \;,\; a = 1,2,3 \;. \tag{4.43}$$

Then, from (4.36), (4.37a), (4.40) and (4.35) we get

$$S_F = \tfrac{1}{2}\int_{X_M}\mathrm{v}_M \,(\boldsymbol{\psi}_0^* \,,\, \boldsymbol{\psi}_0^T\sigma^2) \begin{pmatrix} \mathrm{i}\tilde{\sigma}^\mu(\partial_\mu + \tilde{\pi}(A_\mu{+}A''_\mu)) \;;\; -\overline{\tilde{\pi}(\tilde{\boldsymbol{\Phi}}{+}\tilde{\boldsymbol{\Xi}}{+}\tilde{\boldsymbol{\Upsilon}})} \\ -\tilde{\pi}(\tilde{\boldsymbol{\Phi}}{+}\tilde{\boldsymbol{\Xi}}{+}\tilde{\boldsymbol{\Upsilon}})^* \;;\; \mathrm{i}\sigma^\mu(\partial_\mu + \overline{\tilde{\pi}(A_\mu{+}A''_\mu)}) \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_0 \\ \sigma^2\overline{\boldsymbol{\psi}_0} \end{pmatrix} . \tag{4.44}$$

This formula can be further simplified, because we have

$$
\begin{aligned}
\int_{X_M}\mathrm{v}_M\;\boldsymbol{\psi}_0^T\sigma^2\mathrm{i}\sigma^\mu(\partial_\mu+\overline{\tilde{\pi}(A_\mu{+}A''_\mu)})\sigma^2\overline{\boldsymbol{\psi}_0} &= \int_{X_M}\mathrm{v}_M\;\boldsymbol{\psi}_0^T\mathrm{i}(\tilde{\sigma}^\mu)^T(\partial_\mu+\overline{\tilde{\pi}(A_\mu{+}A''_\mu)})\overline{\boldsymbol{\psi}_0}\\
&= \int_{X_M}\mathrm{v}_M\;\big((-\mathrm{i}\partial_\mu\boldsymbol{\psi}_0^T)(\tilde{\sigma}^\mu)^T\overline{\boldsymbol{\psi}_0}+\boldsymbol{\psi}_0^T(\tilde{\sigma}^\mu)^T(-\mathrm{i}\tilde{\pi}(A_\mu{+}A''_\mu))^T\overline{\boldsymbol{\psi}_0}\big)\\
&= \int_{X_M}\mathrm{v}_M\;\boldsymbol{\psi}_0^*\mathrm{i}\tilde{\sigma}^\mu(\partial_\mu+\tilde{\pi}(A_\mu{+}A''_\mu))\boldsymbol{\psi}_0\;.
\end{aligned}
\tag{4.45}
$$

Here, we have partially integrated and made use of $\tilde{\pi}(A_\mu{+}A''_\mu)=-\tilde{\pi}(A_\mu{+}A''_\mu)^*$. In the last step we took into account that in quantum mechanics the fields $\boldsymbol{\psi}_0$ are annihilation operators and the fields $\overline{\boldsymbol{\psi}_0}$ creation operators. In normal ordered products, all creation operators must stand on the left of all annihilation operators. This means that in (4.45) we have to exchange $\boldsymbol{\psi}_0$ and $\overline{\boldsymbol{\psi}_0}$. But since they represent fermions, which anticommute, this change of order gives a minus sign. Now, (4.44) takes the form

$$
S_F=\int_{X_M}\mathrm{v}_M\;\big(\boldsymbol{\psi}_0^*\mathrm{i}\tilde{\sigma}^\mu(\partial_\mu+\tilde{\pi}(A_\mu{+}A''_\mu))\boldsymbol{\psi}_0-\tfrac{1}{2}(\boldsymbol{\psi}_0^*\tilde{\pi}(\tilde{\boldsymbol{\Phi}}{+}\tilde{\boldsymbol{\Xi}}{+}\tilde{\boldsymbol{\Upsilon}})\sigma^2\overline{\boldsymbol{\psi}_0}+\mathrm{h.c})\big)\;,
\tag{4.46}
$$

where h.c denotes the Hermitian conjugate of the preceding term, without change of signs when exchanging fermion fields. For $\boldsymbol{\psi}_0\in L^2(X_M)\otimes\mathbb{C}^2\otimes\mathbb{C}^{48}$ we choose the following parametrization:

$$
\begin{aligned}
\boldsymbol{\psi}_0 &= \big(u_L^r\,,\,u_L^b\,,\,u_L^g\,,\,d_L^r\,,\,d_L^b\,,\,d_L^g\,,\sigma^2\bar{d}_R^r\,,\sigma^2\bar{d}_R^b\,,\sigma^2\bar{d}_R^g\,,\,\sigma^2\bar{\nu}_R\,,\\
&\qquad -\sigma^2\bar{u}_R^r\,,\,-\sigma^2\bar{u}_R^b\,,\,-\sigma^2\bar{u}_R^g\,,\,-e_L\,,\,\nu_L\,,\,\sigma^2\bar{e}_R\big)^t,\\
\sigma^2\bar{\boldsymbol{\psi}}_0 &= \big(\sigma^2\bar{u}_L^r\,,\sigma^2\bar{u}_L^b\,,\sigma^2\bar{u}_L^g\,,\,\sigma^2\bar{d}_L^r\,,\,\sigma^2\bar{d}_L^b\,,\,\sigma^2\bar{d}_L^g\,,-d_R^r\,,-d_R^b\,,-d_R^g\,,-\nu_R\,,\\
&\qquad u_R^r\,,\,u_R^b\,,\,u_R^g\,,-\sigma^2\bar{e}_L\,,\,\sigma^2\bar{\nu}_L\,,-e_R\big)^t,
\end{aligned}
\tag{4.47}
$$

where $u_L^r,\dots,e_R\in L^2(X_M)\otimes\mathbb{C}^2\otimes\mathbb{C}^3$ and ^{t} means transposition only of the row, without transposing the matrix elements.

Inserting the matrix structures of (4.16d), (4.16e), (4.18) and (4.19) into formulae (4.37b), it is straightforward to write down the explicit formula for the fermionic action (4.46). Here, one must insert the explicit form [12] of the embeddings $\pi_{10},\pi_{10,10},\pi_{10,5}$ and $\pi_{5,1}$. The transformation (4.29e) requires some care. Let us derive the coefficients of P,Z,Z' corresponding to the left electron. From (4.37b), (4.19a) and (4.33) we find for $\lambda_4\ll 1$ in good approximation

$$
\begin{aligned}
\pi_{e_L}(A_\mu+A''_\mu) &\to -\mathrm{i}\tfrac{g_0}{2}(W_\mu^3-\sqrt{\tfrac{3}{5}}A'-\tfrac{3}{2}\sqrt{\tfrac{2}{5}}\tilde{A}_\mu)\\
&= -\tfrac{1}{2\cos(\theta_W-2\theta'_W)}\mathrm{i}g_0\tilde{Z}_\mu-\mathrm{i}\tilde{e}\tilde{Z}'_\mu+\mathrm{i}e\tilde{P}_\mu\;,\qquad\text{where}\\
\tilde{P}_\mu &:= P_\mu-\tan(\theta_W{-}2\theta'_W)Z_\mu+(\tfrac{4}{\sqrt{15}}+\tfrac{12}{5}\theta'_W)Z'_\mu\;,\\
\tilde{Z}_\mu &:= Z_\mu-\tfrac{1}{2}(1+2\sqrt{15}\theta'_W)Z'_\mu\;,\\
\tilde{Z}'_\mu &:= Z'_\mu+4\theta'_W\tan\theta_W Z_\mu\;,\\
e &:= \sin\theta_W g_0\;,\qquad \tilde{e}:=\cos\theta_W g_0\;.
\end{aligned}
\tag{4.48}
$$

Moreover, we express $\Phi_0, \Phi_g, \Xi_A, \dots, \Xi_c, \Upsilon_A, \dots, \Upsilon_g$ in terms of the physical Higgs bosons $\phi_0, \phi_g, \xi_A, \dots, \xi_c, \upsilon_A, \dots, \upsilon_g$, see (4.16d), (4.18) and (4.30). Then we arrive at the following formula for the fermionic Lagrangian:

$$S_F = \int_{X_M} \mathrm{v}_M \,(\mathcal{L}_q + \mathcal{L}_\ell + \mathcal{L}_m + \mathcal{L}_x + \mathcal{L}_h + \mathcal{L}'_h + \mathcal{L}''_h)\;, \qquad \text{where} \tag{4.49a}$$

$$\begin{aligned}
\mathcal{L}_q &= \Big(\boldsymbol{u}_L^*\,,\,\boldsymbol{d}_L^*\Big)\Big(\tilde{\sigma}^\mu \begin{pmatrix} \begin{bmatrix} \mathrm{i}\partial_\mu - \frac{g_0}{2}\mathbf{G}_\mu \\ -(\frac{g_0}{2\cos(\theta_W-2\theta'_W)}\tilde{Z}_\mu \\ +\frac{2}{3}e\tilde{P}_\mu - \frac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbb{1}_3 \end{bmatrix} & -\frac{g_0}{2}(W^1_\mu - \mathrm{i}W^2_\mu)\mathbb{1}_3 \\ -\frac{g_0}{2}(W^1_\mu + \mathrm{i}W^2_\mu)\mathbb{1}_3 & \begin{bmatrix} \mathrm{i}\partial_\mu - \frac{g_0}{2}\mathbf{G}_\mu \\ -(-\frac{g_0}{2\cos(\theta_W-2\theta'_W)}\tilde{Z}_\mu \\ -\frac{1}{3}e\tilde{P}_\mu - \frac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbb{1}_3 \end{bmatrix} \end{pmatrix} \otimes \mathbb{1}_3\Big)\begin{pmatrix}\boldsymbol{u}_L \\ \boldsymbol{d}_L\end{pmatrix} \\
&\quad + \boldsymbol{u}_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - \tfrac{g_0}{2}\mathbf{G}_\mu - (\tfrac{2}{3}e\tilde{P}_\mu - \tfrac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbb{1}_3)\otimes \mathbb{1}_3\big)\boldsymbol{u}_R \\
&\quad + \boldsymbol{d}_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - \tfrac{g_0}{2}\mathbf{G}_\mu - (-\tfrac{1}{3}e\tilde{P}_\mu - \tfrac{1}{3}\tilde{e}\tilde{Z}'_\mu)\mathbb{1}_3)\otimes \mathbb{1}_3\big)\boldsymbol{d}_R\,,
\end{aligned} \tag{4.49b}$$

$$\begin{aligned}
\mathcal{L}_\ell &= \Big(\nu_L^*\,,\,e_L^*\Big)\Big(\tilde{\sigma}^\mu \begin{pmatrix} \mathrm{i}\partial_\mu - (\frac{g_0}{2\cos(\theta_W-2\theta'_W)}\tilde{Z}_\mu + \tilde{e}\tilde{Z}'_\mu) & -\frac{g_0}{2}(W^1_\mu - \mathrm{i}W^2_\mu) \\ -\frac{g_0}{2}(W^1_\mu + \mathrm{i}W^2_\mu) & \begin{bmatrix} \mathrm{i}\partial_\mu - (-\frac{g_0}{2\cos(\theta_W-2\theta'_W)}\tilde{Z}_\mu \\ -e\tilde{P}_\mu + \tilde{e}\tilde{Z}'_\mu) \end{bmatrix} \end{pmatrix} \otimes \mathbb{1}_3\Big)\begin{pmatrix}\nu_L \\ e_L\end{pmatrix} \\
&\quad + \nu_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - \tilde{e}\tilde{Z}'_\mu)\otimes \mathbb{1}_3\big)\nu_R \\
&\quad + e_R^*\big(\sigma^\mu(\mathrm{i}\partial_\mu - (-e\tilde{P}_\mu + \tilde{e}\tilde{Z}'_\mu))\otimes \mathbb{1}_3\big)e_R\,,
\end{aligned} \tag{4.49c}$$

$$\begin{aligned}
\mathcal{L}_m &= \big(- \boldsymbol{d}_L^*(\mathbb{1}_3 \otimes (M_d + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_d) - \tfrac{g_0}{\sqrt{8\mu_3}}(\xi^0_F)^* \otimes M_N)\boldsymbol{d}_R \\
&\quad - e_L^*(M_e + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_e)e_R \\
&\quad - \boldsymbol{u}_L^*(\mathbb{1}_3 \otimes (M_u + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_{\tilde{u}}) + \tfrac{g_0}{4\sqrt{\mu_2}}\upsilon_A \otimes M_{\tilde{n}})\boldsymbol{u}_R \\
&\quad - \nu_L^*(M_n^T + \tfrac{g_0}{\sqrt{8\mu_1}}\phi_0 M_{\tilde{u}}^T - \tfrac{3g_0}{4\sqrt{6\mu_2}}(\upsilon_0 + \mathrm{i}\upsilon_{45})M_{\tilde{n}}^T)\nu_R \\
&\quad - \boldsymbol{d}_L^*(\tfrac{g_0}{4\sqrt{\mu_2}}\upsilon_B \otimes M_{\tilde{n}})\boldsymbol{u}_R - e_L^*(\tfrac{3g_0}{4\sqrt{6\mu_2}}(\upsilon_{18} + \mathrm{i}\upsilon_{63})\otimes M_{\tilde{n}}^T)\nu_R \\
&\quad + \boldsymbol{u}_L^*((\xi^0_E)^* \otimes M_N)\boldsymbol{d}_R - \tfrac{1}{2}\nu_R^T\sigma_2(M_N + \tfrac{g_0}{\sqrt{8\mu_3}}\xi_0 M_N)\nu_R\big) + \text{h.c}\,,
\end{aligned} \tag{4.49d}$$

$$\begin{aligned}
\mathcal{L}_x &= \tfrac{g_0}{2}\big(- \boldsymbol{u}_L^*(\tilde{\sigma}^\mu\sigma^2\epsilon(\bar{X}_\mu)\otimes \mathbb{1}_3)\bar{\boldsymbol{d}}_R - \boldsymbol{d}_L^*(\tilde{\sigma}^\mu\sigma^2\epsilon(\bar{Y}_\mu)\otimes \mathbb{1}_3)\bar{\boldsymbol{d}}_R \\
&\quad + \boldsymbol{u}_L^*(\tilde{\sigma}^\mu\sigma^2 Y_\mu \otimes \mathbb{1}_3)\bar{\nu}_R - \boldsymbol{d}_L^*(\tilde{\sigma}^\mu\sigma^2 X_\mu \otimes \mathbb{1}_3)\bar{\nu}_R \\
&\quad - \nu_L^*(\tilde{\sigma}^\mu\sigma^2 Y_\mu^T \otimes \mathbb{1}_3)\bar{\boldsymbol{u}}_R + e_L^*(\tilde{\sigma}^\mu\sigma^2 X_\mu^T \otimes \mathbb{1}_3)\bar{\boldsymbol{u}}_R\big) + \text{h.c}\,,
\end{aligned} \tag{4.49e}$$

$$\begin{aligned}
\mathcal{L}_h &= \tfrac{g_0}{\sqrt{8\mu_1}}\big(- \boldsymbol{u}_L^*\{\sigma^2\epsilon(\bar{\phi}_g)\otimes M_d\}\bar{\boldsymbol{d}}_L + \nu_R^T\{\sigma^2\phi_g^* \otimes M_d\}\boldsymbol{d}_R \\
&\quad - \boldsymbol{u}_L^*\{\sigma^2\phi_g \otimes M_{\tilde{u}}\}\bar{e}_L + \boldsymbol{d}_L^*\{\sigma^2\phi_g \otimes M_{\tilde{u}}\}\bar{\nu}_L \\
&\quad + \boldsymbol{d}_R^T\{\sigma^2\epsilon(\phi_g)\otimes M_{\tilde{u}}\}\boldsymbol{u}_R - \boldsymbol{u}_R^T\{\sigma^2\bar{\phi}_g \otimes M_e\}e_R\big) + \text{h.c}\,,
\end{aligned} \tag{4.49f}$$

$$\begin{aligned}
\mathcal{L}'_h &= \tfrac{g_0}{4\sqrt{\mu_2}}\big(\boldsymbol{u}_L^*(\sigma^2(\upsilon_a + \upsilon_b)\otimes M_{\tilde{n}})\bar{e}_L - \boldsymbol{d}_L^*(\sigma^2(\upsilon_a - \upsilon_b)\otimes M_{\tilde{n}})\bar{\nu}_L \\
&\quad - \boldsymbol{d}_R^T(\sigma^2(\upsilon_C - \epsilon(\upsilon_a))\otimes M_{\tilde{n}})\boldsymbol{u}_R - \boldsymbol{u}_L^*(\sigma^2\upsilon_c \otimes M_{\tilde{n}})\bar{\nu}_L + \boldsymbol{d}_L^*(\sigma^2\upsilon_d \otimes M_{\tilde{n}})\bar{e}_L \\
&\quad - e_L^*(\upsilon_e^* \otimes M_{\tilde{n}}^T)\boldsymbol{d}_R + \nu_L^*(\upsilon_f^* \otimes M_{\tilde{n}}^T)\boldsymbol{d}_R - \nu_R^T(\sigma^2\upsilon_g^* \otimes M_{\tilde{n}})\boldsymbol{u}_R\big) + \text{h.c}\,,
\end{aligned} \tag{4.49g}$$

$$\begin{aligned}
\mathcal{L}''_h &= \tfrac{g_0}{\sqrt{8\mu_3}}\big(- \tfrac{1}{2}\boldsymbol{u}_L^*(\sigma^2\overline{\xi_A}\otimes M_N)\bar{\boldsymbol{u}}_L - \boldsymbol{u}_L^*(\sigma^2(\overline{\xi_D} - \tfrac{1}{2}\varepsilon(\overline{\xi_c}))\otimes M_N)\bar{\boldsymbol{d}}_L \\
&\quad + \boldsymbol{u}_L^*(\xi_a \otimes M_N)\nu_R - \tfrac{1}{2}\boldsymbol{d}_L^*(\sigma^2\overline{\xi_B}\otimes M_N)\bar{\boldsymbol{d}}_L + \boldsymbol{d}_L^*(\xi_b \otimes M_N)\nu_R \\
&\quad + \tfrac{1}{2}\boldsymbol{d}_R^T(\sigma_2\xi_C \otimes M_N)\boldsymbol{d}_R + \boldsymbol{d}_R^T(\sigma_2\overline{\xi_c}\otimes M_N)\nu_R\big) + \text{h.c}\,.
\end{aligned} \tag{4.49h}$$

The Lagrangian $\mathcal{L}_q$ contains the kinetic terms and the strong and electroweak interactions of quarks. The Lagrangian $\mathcal{L}_\ell$ contains the kinetic terms and

electroweak interactions of leptons. The Lagrangian $\mathcal{L}_m$ contains the mass terms of the fundamental fermions and their interactions with the Higgs fields $\phi_0, \xi_E^0, \xi_F^0, \xi_0, \upsilon_A$ and υ_B. The masses of the u, c, t–quarks, the d, s, b–quarks and the e, μ, τ–leptons are the eigenvalues of M_u, M_d and M_e. The mass Lagrangian of the neutrino sector is given by

$$-\tfrac{1}{2}\big(-\nu_L^*\,,\, \nu_R^T\sigma_2\big) \begin{pmatrix} 0 & -M_n \\ -M_n & M_N \end{pmatrix} \begin{pmatrix} \sigma_2\bar{\nu}_L \\ \nu_R \end{pmatrix} + \text{h.c}\;. \tag{4.50}$$

The diagonalization of the mass matrix occurring in (4.50) yields the masses of the neutrinos. The mixing angles are small for $\|M_N\| \gg \|M_n\|$. In this case, the left–handed neutrinos receive a mass of the order $\frac{\|M_n\|^2}{2\|M_N\|}$ and the right–handed neutrinos a mass of the order $\frac{1}{2}\|M_N\|$. Thus, for $\|M_n\|$ being of the order of the mass of the top quark and $\|M_N\|$ being of the order of the unification scale, we obtain very low masses for the left–handed neutrinos, which is compatible with experiments (seesaw mechanisms). Moreover, the matrices M_u, M_d, M_e, M_n and M_N contain mixing angles between the fermions, which constitute generalized Kobayashi–Maskawa matrices. Finally, the Lagrangians $\mathcal{L}_x, \mathcal{L}_h, \mathcal{L}_h'$ and $\mathcal{L}_h''$ describe the coupling of the fundamental fermions to the X and Y leptoquarks, the Higgs bosons ϕ_g and the remaining Higgs bosons υ_i and ξ_i, respectively. All terms of these Lagrangians contribute to the proton decay.

Observe that the Lagrangians $\mathcal{L}_q$ and $\mathcal{L}_\ell$ differ from the corresponding Lagrangians of the standard model in two aspects: First, there occurs the massive gauge field Z', which of course is not a terrible problem if its mass is sufficiently large. Second, the universal Weinberg angle θ_W of the standard model is modified by angles of the order θ_W'. However, this angle θ_W' is extremely small if $m_{Z'}$ is very large against m_Z. This means that experiments will certainly not detect θ_W'.

5. The Masses of Yang–Mills and Higgs Fields

The final step is to compute the boson masses. For that purpose we must compute the parameters $\mu^i, \tilde{\mu}^i, \breve{\mu}^i, \hat{\mu}^i$ of the Higgs potential (C.1), which depend according to Appendix B on the mass matrices occurring in the generalized Dirac operator $\mathcal{M}$. We have found in Section 4.4 that the eigenvalues

$$\begin{aligned} &\text{of } M_uM_u^* \text{ are } m_u^2, m_c^2, m_t^2\;, \qquad && \text{of } M_dM_d^* \text{ are } m_d^2, m_s^2, m_b^2\;, \\ &\text{of } M_eM_e^* \text{ are } m_e^2, m_\mu^2, m_\tau^2\;, \end{aligned} \tag{5.1}$$

referring to the usual names of the fermions. By unitary transformations we can achieve that M_u is diagonal,

$$M_u = \operatorname{diag}(m_u, m_c, m_t)\;. \tag{5.2a}$$

It is necessary to make several assumptions to simplify the calculation: Since the Kobayashi–Maskawa matrix between M_u and M_d is approximately the identity matrix, let us assume

$$M_d = \operatorname{diag}(m_d, m_s, m_b)\;. \tag{5.2b}$$

The experimental data show that m_t is much bigger than all other eigenvalues. Among the remaining eigenvalues we neglect all but m_b^2 and m_τ^2 . For simplicity we also neglect m_τ^2 against m_b^2 , although this is not completely justified. Unfortunately, there are no experimental values for the matrix M_n . Therefore, we can only estimate its contribution: We assume that in the case (5.2a) we have

$$M_n = \mathrm{diag}(0, 0, \mathrm{e}^{\mathrm{i}\chi} m_n) \;. \tag{5.2c}$$

Quantum corrections suggest that m_n is of the order m_t . Using (3.27) we find for (B.1) approximately

$$\begin{aligned} \mu_1 &= \tfrac{1}{8} m_b^2 + \tfrac{1}{96}(9m_t^2 + 6m_t m_n \cos\chi + m_n^2) + \tfrac{1}{24} m_\tau^2 \;, \\ \mu_2 &= \tfrac{1}{384}(m_t^2 - 2m_t m_n \cos\chi + m_n^2) \;, \end{aligned} \tag{5.3}$$

which yields according to (4.33) for the mass m_W of the W boson

$$m_W^2 = \tfrac{1}{4}(m_t^2 + m_b^2 + \tfrac{1}{3} m_n^2 + \tfrac{1}{3} m_\tau^2) \;. \tag{5.4}$$

The comparison with the experimental values for m_t and m_W requires that m_n is small against m_t . Thus, we shall neglect m_n against m_t whenever this is possible.

Since (at energies accessible at present) the standard model is in excellent agreement with experiments, the parameter $\mu_3 \sim \mathrm{tr}(M_N M_N^*)$ must be very large, see Sections 4.3 and 4.4. We choose the parametrization

$$M_N = m_N \, U \, \mathrm{diag}(\sin\theta_N \cos\phi_N \,, \, \sin\theta_N \sin\phi_N \,, \, \cos\theta_N) U^T \;, \tag{5.5}$$

for $U \in \mathrm{U}(3)$, where the parameter $m_N \gg m_t$ determines the mass scale.

The mass of the X and Y bosons must be very large in order to suppress the proton decay. This could be achieved by a sufficiently large μ_3 , however, there are also Higgs bosons which induce an insufficient lifetime for the proton if μ_0 is too small. Therefore, we must demand

$$\max(\mathrm{tr}(M_{10} M_{10}^*) \,, \, \mathrm{tr}(M_5 M_5^*)) \gg \mathrm{tr}(M_u M_u^*) \;. \tag{5.6}$$

We put[3]

$$M_{10} = M \mathbb{1}_3 + m_{10} \;, \quad M_5 = M \mathbb{1}_3 + m_5 \;, \quad M \in \mathbb{R} \;, \tag{5.7}$$

where $m_{10}, m_5 \in \mathrm{M}_3\mathbb{C}$ are perturbations, which we consider for the time being as small against $M\mathbb{1}_3$. Thus, we obtain for (B.1) approximately

$$\mu_0 = \tfrac{1}{4} M^2 \;, \qquad \mu_1 = \tfrac{3}{32} m_t^2 \;, \qquad \mu_2 = \tfrac{1}{384} m_t^2 \;, \qquad \mu_3 = \tfrac{1}{48} m_N^2 \;. \tag{5.8}$$

Inserting the leading approximation (5.7) into the quadratic terms (C.1) of the Higgs potential, we can distinguish linear combinations of μ^{i} to μ^{t} that do not depend on M . It turns out that the following combinations are essential:

$$\begin{aligned} &\tfrac{1}{4}\mu^{\mathrm{i}} + \tfrac{1}{4}\mu^{\mathrm{j}} + \tfrac{1}{4}\mu^{\mathrm{k}} + \tfrac{1}{4}\mu^{\mathrm{l}} - \tfrac{1}{4}\mu^{\mathrm{m}} + \tfrac{1}{4}\mu^{\mathrm{n}} - \tfrac{1}{2}\mu^{\mathrm{p}} + \tfrac{1}{4}\mu^{\mathrm{r}} - \tfrac{1}{4}\mu^{\mathrm{t}} = \tfrac{1}{2}\,\mathrm{tr}(\tilde{M}_u \tilde{M}_u^* + \hat{M}_u \hat{M}_u^*) =: \tilde{\lambda}_1^2 m_t^4 \;, \\ &\tfrac{1}{4}\mu^{\mathrm{i}} + \tfrac{1}{4}\mu^{\mathrm{j}} + \tfrac{9}{4}\mu^{\mathrm{k}} + \tfrac{9}{4}\mu^{\mathrm{l}} - \tfrac{1}{4}\mu^{\mathrm{m}} - \tfrac{3}{4}\mu^{\mathrm{n}} + \tfrac{3}{2}\mu^{\mathrm{p}} - \tfrac{3}{4}\mu^{\mathrm{r}} - \tfrac{9}{4}\mu^{\mathrm{t}} \\ &\qquad = \tfrac{1}{2}\,\mathrm{tr}(\tilde{M}_n \tilde{M}_n^* + \hat{M}_n \hat{M}_n^*) =: \tilde{\lambda}_2^2 m_t^2 m_n^2 \;, \end{aligned}$$

[3] The choice $M_{10} = (M\mathbb{1}_3 + m_{10})$, $M_5 = \mathrm{e}^{\mathrm{i}\chi_0}(M\mathbb{1}_3 + m_5)$ yields the same results.

$$\begin{aligned}&\tfrac{1}{2}\mu^{\mathrm{i}}+\tfrac{1}{2}\mu^{\mathrm{j}}-\tfrac{3}{2}\mu^{\mathrm{k}}-\tfrac{3}{2}\mu^{\mathrm{l}}-\tfrac{1}{2}\mu^{\mathrm{m}}-\tfrac{1}{2}\mu^{\mathrm{n}}+\mu^{\mathrm{p}}-\tfrac{1}{2}\mu^{\mathrm{r}}+\tfrac{3}{2}\mu^{\mathrm{t}} \\ &\qquad = \tfrac{1}{2}\,\mathrm{tr}(\tilde{M}_u\tilde{M}_n^*+\hat{M}_u\hat{M}_n^*+\tilde{M}_n\tilde{M}_u^*+\hat{M}_n\hat{M}_u^*) =: 2\tilde{\lambda}_3^2 m_t^3 m_n\cos\chi\;, \\ &\mu^{\mathrm{o}}-2\mu^{\mathrm{q}}+\mu^{\mathrm{s}} = \tfrac{1}{2}\mathrm{i}\;\mathrm{tr}(\tilde{M}_u\tilde{M}_n^*+\hat{M}_u\hat{M}_n^*-\tilde{M}_n\tilde{M}_u^*-\hat{M}_n\hat{M}_u^*) =: 2\tilde{\lambda}_4^2 m_t^3 m_n\sin\chi\;,\end{aligned} \tag{5.9a}$$

where

$$\begin{aligned}\tilde{M}_u &= m_{10}M_u - M_u m_5\;, & \hat{M}_u &= m_{10}^*M_u - M_u m_5^*\;, \\ \tilde{M}_n &= m_{10}M_n - M_n m_5\;, & \hat{M}_n &= m_{10}^*M_n - M_n m_5^*\;,\end{aligned} \tag{5.9b}$$

see (3.27) and (B.2). Within our assumptions (5.2) we have

$$\tilde{\lambda}_1=\tilde{\lambda}_2=\tilde{\lambda}_3=\tilde{\lambda}_4\equiv\lambda<\tfrac{M}{m_t}\;. \tag{5.9c}$$

The matrices M_{10}' and M_5' enter the matrices (A.1) only quadratically. Neglecting quadratic terms in m_{10} and m_5 we have

$$M_i^2=\mathrm{diag}(M^2\mathbb{1}_3+M(m_i+m_i^*)\,,\,M^2\mathbb{1}_3+M(m_i+m_i^*))\;,\quad i=10,\,5\;.$$

Thus, we may assume $m_{10}=m_{10}^*$ and $m_5=m_5^*$. Moreover, we may assume $\mathrm{tr}(m_{10})=0$, because the transformation $m_5\mapsto m_5+\nu\mathbb{1}_3$ and $m_{10}\mapsto m_{10}+\nu\mathbb{1}_3$, for $\nu\in\mathbb{R}$, leaves the matrices M_{aa}^i and $\hat{M}_{aa}^i$ invariant. Therefore, we make the ansatz ($\nu_i^j\in\mathbb{R}\,,\ j\in\{10,5\}$)

$$m_j=\begin{pmatrix}\sqrt{\frac{1}{3}}\nu_0^j+\nu_3^j+\sqrt{\frac{1}{3}}\nu_8^j & \nu_1^j-\mathrm{i}\nu_2^j & \nu_4^j-\mathrm{i}\nu_5^j \\ \nu_1^j+\mathrm{i}\nu_2^j & \sqrt{\frac{1}{3}}\nu_0^j-\nu_3^j+\sqrt{\frac{1}{3}}\nu_8^j & \nu_6^j-\mathrm{i}\nu_7^j \\ \nu_4^j+\mathrm{i}\nu_5^j & \nu_6^j+\mathrm{i}\nu_7^j & \sqrt{\frac{1}{3}}\nu_0^j-\sqrt{\frac{4}{3}}\nu_8^j\end{pmatrix},\quad j=10,5\;,\ \nu_0^{10}\equiv 0\;. \tag{5.10}$$

We introduce the abbreviations

$$\begin{aligned}&\cos^4\theta_N+\sin^4\theta_N(\cos^4\phi_N+\sin^4\phi_N)\equiv\tfrac{1}{3}(1+2\cos^2\chi_N)\;, \\ &\nu_{10}^2=2\textstyle\sum_{i=1}^8(\nu_i^{10})^2\;,\qquad (\nu_1^{10})^2+(\nu_2^{10})^2=\tfrac{1}{\sqrt{2}}\nu_{10}\sin\tilde{\chi}\sin\tilde{\chi}'\cos\tilde{\chi}''\;.\end{aligned} \tag{5.11}$$

For physical reasons we assume

$$M,m_N\gg\lambda m_t,\tilde{\lambda}m_t\gg m_t\gg m_b,m_n,m_\tau\;, \tag{5.12}$$

where $\tilde{\lambda}^2m_t^2:=\frac{8}{5}(\frac{9}{2}\nu_{10}^2+\frac{9}{16}(\nu_0^5)^2+\nu_5^2)$. Inserting (5.2), (5.10), (5.2c) and (5.5) into (B.2), (B.3), (B.4) and (B.5) and this result into the Higgs potential (C.1), we find that – apart from the combinations (5.9a) – only the following parameters are relevant in leading approximation:

$$\begin{aligned}&\mu^{\mathrm{b}}=\tfrac{351}{160}m_t^4\;, && \mu^{\mathrm{c}}=\tfrac{13}{7680}m_t^4\;, \\ &\mu^{\mathrm{f}}=\tfrac{39}{320}m_t^4\;, && \mu^{\mathrm{g}}=12M^2m_b^2\;, \\ &\mu^{\mathrm{h}}=\tfrac{1}{2}m_b^2\nu_{10}^2\sin^2\tilde{\chi}\sin^2\tilde{\chi}'\sin^2\tilde{\chi}'' && \mu^{\mathrm{i}}=\mu^{\mathrm{j}}=\tfrac{1}{2}\mu^{\mathrm{m}}=\tfrac{9}{4}M^2m_t^2\;, \\ &\mu^{\mathrm{k}}=\mu^{\mathrm{l}}=\tfrac{1}{2}\mu^{\mathrm{t}}=\tfrac{1}{4}M^2m_t^2\;, && \mu^{\mathrm{n}}=\mu^{\mathrm{p}}=\mu^{\mathrm{r}}=\tfrac{3}{2}M^2m_t^2\;, \\ &\breve{\mu}^{\mathrm{a}}=\tfrac{1}{120}m_N^4(1+16\cos^2\chi_N)\;, && \breve{\mu}^{\mathrm{c}}=\tfrac{3}{10}m_N^2m_t^2(|1+2\cos\chi_N\cos\hat{\chi}|-\tfrac{5}{4})\;, \\ &\breve{\mu}^{\mathrm{d}}=\tfrac{1}{120}m_N^2m_t^2(|1+2\cos\chi_N\cos\hat{\chi}|-\tfrac{5}{4})\;, && \breve{\mu}^{\mathrm{e}}=2M^2m_N^2\;, \\ &\tilde{\mu}^{\mathrm{b}}=\tfrac{1}{176}m_t^4\;, && \tilde{\mu}^{\mathrm{d}}=\tfrac{9}{176}m_t^4\;, \\ &\tilde{\mu}^{\mathrm{k}}=\tfrac{3}{88}m_t^4\;, && \tilde{\mu}^{\mathrm{p}}=\tfrac{1}{192}m_t^4\sin^2\hat{\chi}\;, \\ &\tilde{\mu}^{\mathrm{q}}=\tfrac{3}{64}m_t^4\sin^2\hat{\chi}\;, && \tilde{\mu}^{\mathrm{s}}=\tfrac{1}{32}m_t^4\sin^2\hat{\chi}\;,\end{aligned} \tag{5.13}$$

$$\hat\mu^{\mathrm{a}} = 2m_N^4((1+2\cos^2\chi_N)-\tfrac{1}{4})\cos^2\hat\chi_a\;,\quad \hat\mu^{\mathrm{c}} = \tfrac{1}{88}m_N^2m_t^2(|1+2\cos\chi_N\cos\hat\chi|-1)\;,$$
$$\hat\mu^{\mathrm{e}} = \tfrac{3}{88}m_N^2m_t^2(|1+2\cos\chi_N\cos\hat\chi|-1)\;.$$

The parameters $\hat\chi$ and $\hat\chi_a$ are complicated functions of the mass matrices. Now we find for (4.34c) in tree–level approximation

$$\begin{aligned}
\mathcal{L}_H = {}& \tfrac12 g^{\mu\nu}(\partial_\mu\phi_0'\,\partial_\nu\phi_0' + \partial_\mu\upsilon_0'\,\partial_\nu\upsilon_0' + \partial_\mu\upsilon_{45}\,\partial_\nu\upsilon_{45} \\
&\quad +\partial_\mu\psi_0\,\partial_\nu\psi_0 + \partial_\mu\psi_3'\,\partial_\nu\psi_3' + \partial_\mu\xi_0\,\partial_\nu\xi_0) \\
&\quad -\tfrac12\big(\lambda^2m_t^2\upsilon_0^2 + \tfrac34\lambda^2m_t^2\upsilon_{45}^2 + \tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat\chi_a + \tfrac{1}{54}\cos^2\chi_N\cos^2\hat\chi_a){\psi_3'}^2 \\
&\quad +\tfrac{1}{48M^2}(\tfrac{48}{5}\check\mu^{\mathrm{e}} + \tfrac29\hat\mu^{\mathrm{a}})\psi_0^2 + \tfrac{1}{2m_N^2}(4\check\mu^{\mathrm{a}} + \tfrac{8}{15}\hat\mu^{\mathrm{a}})\xi_0^2 \\
&\quad +(\tfrac{207}{110} + \tfrac29\sin^2\hat\chi)m_t^2{\phi_0'}^2 + \tfrac{1}{8\sqrt{15}Mm_t}(-\tfrac{32}{3}\hat\mu^{\mathrm{c}} + \tfrac{32}{3}\hat\mu^{\mathrm{e}})\psi_0\phi_0' \\
&\quad +\tfrac{1}{\sqrt{24}m_Nm_t}(4\check\mu^{\mathrm{c}} + 48\check\mu^{\mathrm{d}} + \tfrac{64}{5}\hat\mu^{\mathrm{c}} - \tfrac{64}{5}\hat\mu^{\mathrm{e}})\phi_0'\xi_0 - \tfrac{2}{9\sqrt{10}Mm_N}\hat\mu^{\mathrm{a}}\psi_0\xi_0\big) \\
&+\tfrac12 g^{\mu\nu}(\partial_\mu\psi_1'\,\partial_\nu\psi_1' + \partial_\mu\psi_2'\,\partial_\nu\psi_2' + \partial_\mu\upsilon_{18}\,\partial_\nu\upsilon_{18} + \partial_\mu\upsilon_{63}\,\partial_\nu\upsilon_{63}) \\
&\quad -\tfrac12\big(\tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat\chi_a + \tfrac{1}{54}\cos^2\chi_N\cos^2\hat\chi_a)({\psi_1'}^2 + {\psi_2'}^2) + \tfrac34\lambda^2m_t^2(\upsilon_{18}^2 + \upsilon_{63}^2)\big) \\
&+\tfrac12 g^{\mu\nu}(\textstyle\sum_{i=1}^8\partial_\mu\psi_i\,\partial_\nu\psi_i + \sum_{i=1}^8\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=46}^{53}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i \\
&\quad +\textstyle\sum_{i=33}^{40}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=82}^{89}\partial_\mu\xi_i\,\partial_\nu\xi_i) \\
&\quad -\tfrac12\big(\tfrac{m_N^4}{M^2}(\tfrac{1}{144}\cos^2\hat\chi_a + \tfrac{1}{54}\cos^2\chi_N\cos^2\hat\chi_a)\textstyle\sum_{i=1}^8\psi_i^2 \\
&\quad +(\lambda^2m_n^2 + \tfrac{m_b^2\nu_{10}^2}{m_t^2}\sin^2\tilde\chi\sin^2\tilde\chi'\sin^2\tilde\chi'')(\textstyle\sum_{i=1}^8\upsilon_i^2 + \sum_{i=46}^{53}\upsilon_i^2) \\
&\quad +9M^2(\textstyle\sum_{i=33}^{40}\xi_i^2 + \sum_{i=82}^{89}\xi_i^2)\big) \\
&+\tfrac12 g^{\mu\nu}(\textstyle\sum_{i=19}^{26}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=64}^{71}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i \\
&\quad +\textstyle\sum_{i=25}^{32}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=74}^{81}\partial_\mu\xi_i\,\partial_\nu\xi_i) \\
&\quad -\tfrac12\big((\lambda^2m_n^2 + \tfrac{m_b^2\nu_{10}^2}{m_t^2}\sin^2\tilde\chi\sin^2\tilde\chi'\sin^2\tilde\chi'')(\textstyle\sum_{i=19}^{26}\upsilon_i^2 + \sum_{i=64}^{71}\upsilon_i^2) \\
&\quad +9M^2(\textstyle\sum_{i=25}^{32}\xi_i^2 + \sum_{i=74}^{81}\xi_i^2)\big) \\
&+\tfrac12 g^{\mu\nu}(\textstyle\sum_{i=1}^6\partial_\mu\phi_i\,\partial_\nu\phi_i + \sum_{i=9}^{14}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=54}^{59}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i \\
&\quad +\textstyle\sum_{i=30}^{35}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=75}^{80}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=39}^{41}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=84}^{86}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i \\
&\quad +\textstyle\sum_{i=19}^{24}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=68}^{73}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=44}^{46}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=93}^{95}\partial_\mu\xi_i\,\partial_\nu\xi_i \\
&\quad +\textstyle\sum_{i=47}^{49}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=96}^{98}\partial_\mu\xi_i\,\partial_\nu\xi_i) \\
&\quad -\tfrac12\big(M^2(\textstyle\sum_{i=1}^6\phi_i^2 + \sum_{i=9}^{14}\upsilon_i^2 + \sum_{i=54}^{59}\upsilon_i^2 + \sum_{i=30}^{35}\upsilon_i^2 + \sum_{i=75}^{80}\upsilon_i^2) \\
&\quad +4M^2(\textstyle\sum_{i=39}^{41}\upsilon_i^2 + \sum_{i=84}^{86}\upsilon_i^2 + \sum_{i=19}^{24}\xi_i^2 + \sum_{i=68}^{73}\xi_i^2 + \sum_{i=47}^{49}\xi_i^2 + \sum_{i=96}^{98}\xi_i^2) \\
&\quad +(M^2 + m_N^2(\tfrac{1}{12}\cos^2\hat\chi_a + \tfrac29\cos^2\chi_N\cos^2\hat\chi_a))(\textstyle\sum_{i=44}^{46}\xi_i^2 + \sum_{i=93}^{95}\xi_i^2)\big) \\
&+\tfrac12 g^{\mu\nu}(\textstyle\sum_{i=7}^{18}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=56}^{67}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=41}^{43}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=90}^{92}\partial_\mu\xi_i\,\partial_\nu\xi_i \\
&\quad +\textstyle\sum_{i=15}^{17}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=60}^{62}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=42}^{44}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=87}^{89}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i \\
&\quad +\textstyle\sum_{i=36}^{38}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=81}^{83}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i) \\
&\quad -\tfrac12\big(M^2(\textstyle\sum_{i=15}^{17}\upsilon_i^2 + \sum_{i=60}^{62}\upsilon_i^2 + \sum_{i=42}^{44}\upsilon_i^2 + \sum_{i=87}^{89}\upsilon_i^2) \\
&\quad +4M^2(\textstyle\sum_{i=36}^{38}\upsilon_i^2 + \sum_{i=81}^{83}\upsilon_i^2 + \sum_{i=7}^{12}\xi_i^2 + \sum_{i=56}^{61}\xi_i^2) \\
&\quad +16M^2(\textstyle\sum_{i=13}^{18}\xi_i^2 + \sum_{i=62}^{69}\xi_i^2) \\
&\quad +(M^2 + m_N^2(\tfrac{1}{12}\cos^2\hat\chi_a + \tfrac29\cos^2\chi_N\cos^2\hat\chi_a))(\textstyle\sum_{i=41}^{43}\xi_i^2 + \sum_{i=90}^{92}\xi_i^2)\big) \\
&+\tfrac12 g^{\mu\nu}(\textstyle\sum_{i=27}^{29}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=72}^{74}\partial_\mu\upsilon_i\,\partial_\nu\upsilon_i + \sum_{i=1}^{6}\partial_\mu\xi_i\,\partial_\nu\xi_i + \sum_{i=50}^{55}\partial_\mu\xi_i\,\partial_\nu\xi_i) \\
&\quad -\tfrac12\big(4M^2(\textstyle\sum_{i=1}^6\xi_i^2 + \sum_{i=50}^{55}\xi_i^2) + M^2(\sum_{i=27}^{29}\upsilon_i^2 + \sum_{i=72}^{74}\upsilon_i^2)\big)\;.
\end{aligned}\tag{5.14}$$

It remains to find the eigenvalues of the quadratic form[4] determined by the ϕ'_0–ψ_0–ξ_0 sector in (5.14). We use the general result that the smallest (largest) eigenvalue is smaller (larger) than the smallest (largest) diagonal matrix element. This means that the mass of the ϕ'_0 Higgs field is smaller than $\sqrt{\frac{2083}{990}}\, m_t \approx 1.45\, m_t$. We assume $\frac{48}{5}\check{\mu}^{\mathrm{e}} \gg \frac{2}{9}\hat{\mu}^{\mathrm{a}}$, or $M^2 \gg \frac{55}{864} m_N^2$. Then, the large parameter $\check{\mu}^{\mathrm{e}}$ occurring in the coefficient of ψ_0^2 stabilizes the other two eigenvalues near the diagonal matrix elements $\frac{1}{5M^2}\check{\mu}^{\mathrm{e}}$ and $\frac{1}{m_N^2}(2\check{\mu}^{\mathrm{a}} + \frac{4}{15}\hat{\mu}^{\mathrm{a}})$, respectively.

For convenience we list in Table 1 our tree–level predictions for the masses of the Higgs fields and the masses of the gauge fields derived in Section 4.3. We recall that m_t is the mass of the top quark, m_N the mass scale of the right neutrinos and M the grand unification scale, where we have assumed $m_N, M \gg m_t$. Moreover, we have introduced the abbreviation

$$\check{\lambda} = \sqrt{\lambda^2 + \frac{m_b^2 \nu_{10}^2}{m_t^2 m_n^2}} - \lambda \geq 0\ .$$

It is interesting to perform the transformation (4.31) in the Yukawa Larangian $\mathcal{L}_m$ of the fermionic action (4.49). The contribution of the coupling of the ϕ'_0 Higgs field to the fermions takes the form

$$\begin{aligned}
\mathcal{L}_{\phi'_0} &= \big(-\boldsymbol{d}_L^*(\mathbb{1}_3 \otimes (M_d + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi'_0 M_d))\boldsymbol{d}_R - e_L^*(M_e + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi'_0 M_e)e_R \\
&\quad - \boldsymbol{u}_L^*(\mathbb{1}_3 \otimes (M_u + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi'_0 M_u))\boldsymbol{u}_R - \nu_L^*(M_n^T + \tfrac{g_0}{\sqrt{8(\mu_1+12\mu_2)}}\phi'_0 M_n^T)\nu_R\big) \\
&\quad + \mathrm{h.c} \qquad\qquad (5.15) \\
&= \big(-\boldsymbol{d}_L^*(\mathbb{1}_3 \otimes (1 + \tfrac{g_0}{m_t}\phi'_0)M_d)\boldsymbol{d}_R - e_L^*((1 + \tfrac{g_0}{m_t}\phi'_0)M_e)e_R \\
&\quad - \boldsymbol{u}_L^*(\mathbb{1}_3 \otimes (1 + \tfrac{g_0}{m_t}\phi'_0)M_u))\boldsymbol{u}_R - \nu_L^*((1 + \tfrac{g_0}{m_t}\phi'_0)M_n^T)\nu_R\big) + \mathrm{h.c}\ .
\end{aligned}$$

Thus, the Higgs field ϕ'_0 has the same properties as the standard model Higgs field.

All other Higgs fields are too massive to observe. All Higgs and gauge fields with fractional–valued charge lead to proton decay. Without exception they receive a mass of the order of the grand unification scale M, which must be chosen sufficiently large to ensure the observed stability of matter. The mass of the remaining Higgs fields with integer–valued charge is of the order $M, \lambda m_t, \lambda m_n, m_N$ or $\frac{m_N^2}{M}$. These mass scales are situated somewhere between m_t and M. By assumption, m_N and $\frac{m_N^2}{M}$ are very close to M. Moreover, for generic choices of the mass matrices M_{10} and M_5, also λm_t and λm_n are close to M.

6. The SU(5)–Model

Formally, we can derive the SU(5)–model from our flipped SU(5)×U(1)–model discussed so far by the following restrictions and replacements: We put $\tilde{A}_\mu \equiv 0$ ad hoc. Strictly speaking, this step is not allowed within our theory. However, one could imagine a formalism where the connection forms are not $(\Lambda^1 \otimes \mathfrak{r}^0\mathfrak{a}) \oplus$

[4]The corresponding matrix is positive definite by construction. This is not apparent when inserting (5.13), because there are complicated relations between $\chi_N, \hat{\chi}_a, \hat{\chi}$.

Particle	Mass	Particle	Mass
1. The completely neutral Higgs fields:			
ϕ'_0	$(0\dots 1.45)\, m_t$	ξ_0	$(\sqrt{\frac{1}{60}}\dots\sqrt{\frac{7}{4}})m_N$
υ'_0	λm_t	υ_{45}	$\frac{1}{2}\sqrt{3}\lambda m_t$
ψ_0	$\sqrt{\frac{2}{5}}m_N$	ψ'_3	$(0\dots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
2. The colour–neutral Higgs fields of charge ∓ 1:			
$\frac{1}{\sqrt{2}}(\upsilon_{18}\pm \mathrm{i}\upsilon_{63})$	$\frac{1}{2}\sqrt{3}\lambda m_t$	$\frac{1}{\sqrt{2}}(\psi_1\pm \mathrm{i}\psi_2)$	$(0\dots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$
3. The neutral Higgs fields, for $i=0,\dots,7$:			
ψ_{i+1}	$(0\dots\frac{1}{12}\sqrt{\frac{11}{3}})\frac{m_N^2}{M}$		
υ_{i+1}	$(\lambda\dots\lambda{+}\check{\lambda})m_n$	υ_{i+45}	$(\lambda\dots\lambda{+}\check{\lambda})m_n$
ξ_{i+32}	$3M$	ξ_{i+81}	$3M$
4. The Higgs fields of charge ∓ 1, for $i=0\dots 7$:			
$\frac{1}{\sqrt{2}}(\upsilon_{19+i}\pm \mathrm{i}\upsilon_{64+i})$	$(\lambda\dots\lambda{+}\check{\lambda})m_n$	$\frac{1}{\sqrt{2}}(\xi_{25+i}\pm \mathrm{i}\xi_{74+i})$	$3M$
5. The Higgs fields of charge $\mp\frac{1}{3}$, for $i=0,1,2$ and $j=0,\dots,5$:			
$\frac{1}{\sqrt{2}}(\phi_{1+i}\pm \mathrm{i}\phi_{3+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{9+i}\pm \mathrm{i}\upsilon_{54+i})$	M
$\frac{1}{\sqrt{2}}(\upsilon_{12+i}\pm \mathrm{i}\upsilon_{57+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{39+i}\pm \mathrm{i}\upsilon_{84+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{44+i}\pm \mathrm{i}\upsilon_{93+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{47+i}\pm \mathrm{i}\upsilon_{96+i})$	$2M$
$\frac{1}{\sqrt{2}}(\xi_{19+j}\pm \mathrm{i}\upsilon_{68+j})$	$2M$	$\frac{1}{\sqrt{2}}(\upsilon_{30+j}\pm \mathrm{i}\upsilon_{75+j})$	M
6. The Higgs fields of charge $\pm\frac{2}{3}$, for $i=0,1,2$ and $j=0,\dots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{15+i}\pm \mathrm{i}\upsilon_{60+i})$	M	$\frac{1}{\sqrt{2}}(\upsilon_{36+i}\pm \mathrm{i}\upsilon_{81+i})$	$2M$
$\frac{1}{\sqrt{2}}(\upsilon_{42+i}\pm \mathrm{i}\upsilon_{87+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{41+i}\pm \mathrm{i}\upsilon_{90+i})$	M
$\frac{1}{\sqrt{2}}(\xi_{7+j}\pm \mathrm{i}\xi_{56+j})$	$2M$	$\frac{1}{\sqrt{2}}(\xi_{13+j}\pm \mathrm{i}\xi_{62+j})$	$4M$
7. The Higgs fields of charge $\mp\frac{4}{3}$, for $i=0,1,2$ and $j=0,\dots,5$:			
$\frac{1}{\sqrt{2}}(\upsilon_{27+i}\pm \mathrm{i}\upsilon_{72+i})$	M	$\frac{1}{\sqrt{2}}(\xi_{1+j}\pm \mathrm{i}\upsilon_{50+j})$	$2M$
8. The neutral massive gauge fields:			
Z	$\sqrt{\frac{2}{5}}\, m_t$	Z'	$\frac{1}{2}\sqrt{\frac{5}{3}}m_N$
9. The massive gauge fields of charge ± 1:			
$\frac{1}{\sqrt{2}}(W_1\mp \mathrm{i}W_2)$	$\frac{1}{2}m_t$	Weinberg angle: $\sin^2\theta_W=\frac{3}{8}$	
10. The leptoquarks leading to proton decay, for $i=0,1,2$:			
$\frac{1}{\sqrt{2}}(X_{1+i}\mp \mathrm{i}X_{3+i})$	M	charge: $\mp\frac{1}{3}$	
$\frac{1}{\sqrt{2}}(Y_{1+i}\mp \mathrm{i}Y_{3+i})$	M	charge: $\pm\frac{2}{3}$	

TABLE 1. The particle masses for the $\mathrm{SU}(5)\times\mathrm{U}(1)$–model

$(\Lambda^0\gamma^5 \otimes \mathfrak{r}^1\mathfrak{a})$–valued but $(\Lambda^1 \otimes \hat{\pi}(\mathfrak{a})) \oplus (\Lambda^0\gamma^5 \otimes \hat{\pi}(\Omega^1\mathfrak{a}))$–valued. Now, taking the same L–cycle as before, however with $M_N \equiv 0$, we obtain indeed a SU(5)–GUT. The calculation is the same as before. However, since the graded centre $\mathfrak{c}^2\mathfrak{a}$ is not relevant in such a model, we must put $J_3 = 0$ and $\zeta_{A,B,U,V} = 0$ in the factorization procedure of Section 3.6. Moreover, instead of (4.29a) we perform the orthogonal transformation

$$-\begin{pmatrix} Z_\mu \\ P_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W^3_\mu \\ A'_\mu \end{pmatrix}, \qquad \sin\theta_W = \sqrt{\tfrac{3}{8}}\,. \tag{6.1}$$

If we compute the electric charges we find that the labels are unconvenient now, because u describes the d quarks (and vice versa) and ν the electrons (and vice versa). Thus, we must permute the labels $u \leftrightarrow d$ and $\nu \leftrightarrow e$. Then we obtain almost the same form (4.49) for the fermionic action, with the following modifications:

1) We have $M_N \equiv 0$, in particular, the Lagrangian $\mathcal{L}''_h$ and the Higgs fields ξ^0_E, ξ^0_F and ξ_0 are absent.
2) In the Lagrangians $\mathcal{L}_q$ and $\mathcal{L}_\ell$ we have $Z'_\mu \equiv 0$, $\tilde{Z}'_\mu \equiv 0$ and $\theta'_W \equiv 0$.
3) In the Lagrangians $\mathcal{L}_x$, $\mathcal{L}_h$ and $\mathcal{L}'_h$, the fermion labels $\boldsymbol{u}$ and $\boldsymbol{d}$ are exchanged and e and ν are exchanged.
4) The same exchanges occurs for the mass matrices: $M_u \leftrightarrow M_d$, $M_{\tilde{u}} \leftrightarrow M_{\tilde{d}}$, $M_e \leftrightarrow M_n$, $M_{\tilde{n}} \leftrightarrow M_{\tilde{e}}$.

Compared with the SU(5) × U(1)–model one finds [12] identical tree–level predictions for the Weinberg angle, the masses of the W and Z boson and for the Higgs and gauge fields with fractional–valued charge. Again, there is precisely one standard model Higgs field ϕ'_0 whose upper bound for the mass turns out to be $m_{\phi'_0} = 1.32\, m_t$. Moreover, the masses of the remaining Higgs fields with integer–valued charge lie between m_t and M, generically close to M.

7. Conclusion

1) We have succeeded in formulating the flipped SU(5) × U(1)–GUT within the framework of non–associative geometry. We have found interesting tree–level relations between fermionic and bosonic parameters: Given the fermionic parameters (fermion masses and Kobayashi–Maskawa mixing angles) and two 3 × 3–matrices determining the unification scale as input, we were able to compute all bosonic quantities:
   - the occurring multiplets of Higgs fields,
   - the spontaneous symmetry breaking pattern,
   - the masses of all Higgs fields,
   - the masses of all Yang–Mills fields,
   - the Weinberg angle.

   However, since not all input parameters are known, we were forced to be satisfied with estimations for some of the masses.
2) The representation of the U(1)–part of the SU(5) × U(1)–model is not an input but an algebraic consequence of the theory. This U(1)–representation is unique and realized in nature.

3) In the SU(5) × U(1)–model there occur Higgs fields in complex $\underline{5}$–, complex $\underline{50}$–, complex $\underline{45}$– and real $\underline{24}$–plets. After the spontaneous symmetry breaking, there survive 12 Higgs fields of the $\underline{24}$–representation, 7 Higgs fields of the $\underline{5}$–representation, 99 Higgs fields of the $\underline{50}$–representation and 90 Higgs fields of the $\underline{45}$–representation, and 16 gauge fields become massive.
4) There occur three mass scales in the model under consideration:
   - The lowest mass scale is the scale of the fermion masses reaching from the neutrino masses to the mass of the top quark. Moreover, also the electroweak gauge fields Z, W^+, W^- belong to this scale, and – remarkably – one Higgs field as well.
   - The mass of all fields leading to proton decay is of the order of the grand unification scale M.
   - The masses of Higgs fields which do not lead to proton decay lie between the fermions scale and the grand unification scale, generically close to M.
5) There exists precisely one light Higgs field ϕ_0', which has exactly the same properties as the standard model Higgs field. It couples to a fermion of the mass m_f with the coupling constant $g_0 m_f/m_t$. Moreover, it has the same couplings with the intermediate vector bosons Z, W^+, W^- as the standard model Higgs field. The Higgs field ϕ_0' is a certain linear combination of the $\underline{5}$–representation and the $\underline{45}$–representation. This linear combination is the only one which corresponds to a zero mode of the grand unification sector. That the mass of ϕ_0' is generically different from zero is due to the fermion masses. Therefore, the Higgs field ϕ_0' receives a mass of the order of the mass m_t of the top quark: For $m_t = 176\,\mathrm{GeV}$ we have $m_{\phi_0'} \leq 255\,\mathrm{GeV}$. The reason that only an upper bound can be given is the incomplete knowledge of the input parameters. The upper bound is independent of all parameters related to grand unification.
6) The standard model is in perfect agreement with experiment. However, we have shown that the low energy sector of the SU(5) × U(1)–GUT is identical with the standard model. Thus, one must be careful with the extrapolation of the standard model to higher energies.

Appendix A. The Generation Space Matrices

$$
\begin{aligned}
\hat{M}^{10}_{aa} &:= \tfrac{3}{10} M_{10}'^{\,2} + (\tfrac{3}{5}\alpha_A + \zeta_A)\mathbb{1}_6 , & \hat{M}^{10}_{cc} &:= \tfrac{1}{10} M_N' M_N'^{\,*} + (\tfrac{3}{5}\alpha_U + \zeta_U)\mathbb{1}_6 , \\
\hat{M}^{10}_{nn} &= \tfrac{1}{10} M_{\tilde n}' M_{\tilde n}'^{\,*} + (\tfrac{3}{5}\alpha_V + \zeta_V)\mathbb{1}_6 , \\
\hat{M}^{10}_{bb} &:= \tfrac{2}{5} M_{\tilde u}' M_{\tilde u}'^{\,*} + \tfrac{3}{5} M_d' M_d'^{\,*} + (\tfrac{3}{5}\alpha_B + \zeta_B)\mathbb{1}_6 , \\
M^{10}_{aa} &:= M_{10}'^{\,2} + \beta_A \mathbb{1}_6 + 3\delta_A M_{ud}^2 , & M^{10}_{nn} &:= M_{\tilde n}' M_{\tilde n}'^{\,*} + \tfrac{1}{3}\beta_V \mathbb{1}_6 + \delta_V M_{ud}^2 , \\
M^{10}_{cc} &:= M_N' M_N'^{\,*} + \tfrac{1}{3}\beta_U \mathbb{1}_6 + \delta_U M_{ud}^2 , & \check{M}^{10}_{nn} &:= \tfrac{1}{3}\check{\beta}_V \mathbb{1}_6 + \check{\delta}_V M_{ud}^2 , \\
M^{10}_{\{un\}} &:= \tfrac{1}{2}(M_{\tilde u}' M_{\tilde n}'^{\,*} + M_{\tilde n}' M_{\tilde u}'^{\,*}) + \beta_W \mathbb{1}_6 + 3\delta_W M_{ud}^2 , \\
M^{10}_{[un]} &:= \tfrac{1}{2\mathrm{i}}(M_{\tilde u}' M_{\tilde n}'^{\,*} - M_{\tilde n}' M_{\tilde u}'^{\,*}) + \beta_W' \mathbb{1}_6 + 3\delta_W' M_{ud}^2 , \\
\tilde{M}^{10}_{nn} &:= M_{\tilde n}' M_{\tilde n}'^{\,*} + \gamma_V \mathbb{1}_6 + \epsilon_V \tilde{M}_V^2 , & \tilde{M}^{10}_{cc} &:= M_N' M_N'^{\,*} + \gamma_U \mathbb{1}_6 + \epsilon_U \tilde{M}_V^2 , \\
\tilde{M}^{10}_{\{cd\}} &:= \tfrac{1}{2}(M_N' M_d'^{\,*} + M_d' M_N'^{\,*}) + \tilde{\gamma}_U \mathbb{1}_6 + \tilde{\epsilon}_U \tilde{M}_V^2 ,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}^{10}_{[cd]} &:= \tfrac{1}{2\mathrm{i}}(M'_N M'^*_d - M'_d M'^*_N) + \tilde{\gamma}'_U \mathbb{1}_6 + \tilde{\epsilon}'_U \tilde{M}^2_V \;, \\
\tilde{M}^{10}_{\{un\}} &:= \tfrac{1}{2}(M'_{\tilde{u}} M'^*_{\tilde{n}} + M'_{\tilde{n}} M'^*_{\tilde{u}}) + \gamma_W \mathbb{1}_6 + \epsilon_W \tilde{M}^2_V \;, \\
\tilde{M}^{10}_{[un]} &:= \tfrac{1}{2\mathrm{i}}(M'_{\tilde{u}} M'^*_{\tilde{n}} - M'_{\tilde{n}} M'^*_{\tilde{u}}) + \gamma'_W \mathbb{1}_6 + \epsilon'_W \tilde{M}^2_V \;, \\
\hat{M}^5_{aa} &:= \tfrac{1}{5} M'^2_5 + (\tfrac{2}{5}\alpha_A + \zeta_A)\mathbb{1}_6 \;, \qquad \hat{M}^5_{cc} := (\tfrac{2}{5}\alpha_U + \zeta_U)\mathbb{1}_6 \;, \\
\hat{M}^5_{nn} &:= \tfrac{1}{5} M'^*_{\tilde{n}} M'_{\tilde{n}} + (\tfrac{2}{5}\alpha_V + \zeta_V)\mathbb{1}_6 \;, \\
\hat{M}^5_{bb} &:= \tfrac{4}{5} M'^*_{\tilde{u}} M'_{\tilde{u}} + \tfrac{1}{5}\bar{M}'_e M'^T_e + (\tfrac{2}{5}\alpha_B + \zeta_B)\mathbb{1}_6 \;, \\
M^5_{aa} &:= M'^2_5 + \beta_A \mathbb{1}_6 + \delta_A M^2_{en} \;, \qquad \check{M}^5_{nn} := M'^*_{\tilde{n}} M'_{\tilde{n}} + \check{\beta}_V \mathbb{1}_6 + \check{\delta}_V M^2_{en} \;, \\
M^5_{nn} &:= \beta_V \mathbb{1}_6 + \delta_V M^2_{en} \;, \qquad M^5_{cc} := \beta_U \mathbb{1}_6 + \delta_U M^2_{en} \;, \\
M^5_{\{un\}} &:= \tfrac{1}{2}(M'^*_{\tilde{n}} M'_{\tilde{u}} + M'^*_{\tilde{u}} M'_{\tilde{n}}) + \beta_W \mathbb{1}_6 + \delta_W M^2_{en} \;, \\
M^5_{[un]} &:= \tfrac{1}{2\mathrm{i}}(M'^*_{\tilde{n}} M'_{\tilde{u}} - M'^*_{\tilde{u}} M'_{\tilde{n}}) + \beta'_W \mathbb{1}_6 + \delta'_W M^2_{en} \;, \\
\hat{M}^1_{aa} &:= \zeta_A \mathbb{1}_6 \;, \qquad \hat{M}^1_{nn} := \zeta_V \mathbb{1}_6 \;, \\
\hat{M}^1_{cc} &:= \zeta_U \mathbb{1}_6 \;, \qquad \hat{M}^1_{bb} := M'^T_e \bar{M}'_e + \zeta_B \mathbb{1}_6 \;.
\end{aligned}
\tag{A.1}
$$

The real constants $\alpha_A, \dots \zeta_V$ are determined by equation (3.37). The solution is

$$
\begin{aligned}
\alpha_A &= -\tfrac{1}{8}\operatorname{tr}(M'^2_{10}) + \tfrac{1}{24}\operatorname{tr}(M'^2_5) \;, \qquad \alpha_B = -\tfrac{1}{4}\operatorname{tr}(M'_d M'^*_d) + \tfrac{1}{4}\operatorname{tr}(M'_e M'^*_e) \;, \\
\alpha_U &= -\tfrac{1}{24}\operatorname{tr}(M'_N M'^*_N) \;, \qquad \alpha_V = 0 \;, \\
\zeta_A &= \tfrac{1}{32}\operatorname{tr}(M'^2_{10}) - \tfrac{1}{32}\operatorname{tr}(M'^2_5) \;, \qquad \zeta_V = -\tfrac{1}{48}\operatorname{tr}(M'_{\tilde{n}} M'^*_{\tilde{n}}) \;, \\
\zeta_B &= -\tfrac{1}{12}\operatorname{tr}(M'_{\tilde{u}} M'^*_{\tilde{u}}) + \tfrac{1}{16}\operatorname{tr}(M'_d M'^*_d) - \tfrac{7}{48}\operatorname{tr}(M'_e M'^*_e) \;, \\
\zeta_U &= \tfrac{1}{96}\operatorname{tr}(M'_N M'^*_N) \;, \\
\beta_A &= -\tfrac{1}{8}\operatorname{tr}(M'^2_5) - \tfrac{1}{24}\operatorname{tr}(M'^2_{10}) \;, \qquad \delta_A = -\frac{\operatorname{tr}(M'^2_{10} M^2_{ud} + M'^2_5 M^2_{en})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
\beta_U &= -\tfrac{1}{8}\operatorname{tr}(M'_N M'^*_N) \;, \qquad \delta_U = -3\frac{\operatorname{tr}(M'_N M'^*_N M^2_{ud})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
\beta_V &= -\tfrac{1}{8}\operatorname{tr}(M'_{\tilde{n}} M'^*_{\tilde{n}}) \;, \qquad \check{\beta}_V = -\tfrac{1}{8}\operatorname{tr}(M'_{\tilde{n}} M'^*_{\tilde{n}}) \;, \\
\delta_V &= -\frac{\operatorname{tr}(3M'_{\tilde{n}} M'^*_{\tilde{n}} M^2_{ud})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \qquad \check{\delta}_V = -\frac{\operatorname{tr}(M'^*_{\tilde{n}} M'_{\tilde{n}} M^2_{en})}{\operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
\beta_W &= -\tfrac{1}{12}\operatorname{tr}(M'_{\tilde{u}} M'^*_{\tilde{n}} + M'_{\tilde{n}} M'^*_{\tilde{u}}) \;, \qquad \beta'_W = -\tfrac{1}{12\mathrm{i}}\operatorname{tr}(M'_{\tilde{u}} M'^*_{\tilde{n}} - M'_{\tilde{n}} M'^*_{\tilde{u}}) \;, \\
\delta_W &= -\frac{\operatorname{tr}((M'_{\tilde{u}} M'^*_{\tilde{n}} + M'_{\tilde{n}} M'^*_{\tilde{u}}) M^2_{ud} + (M'^*_{\tilde{n}} M'_{\tilde{u}} + M'^*_{\tilde{u}} M'_{\tilde{n}}) M^2_{en})}{2\ \operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
\delta'_W &= -\frac{\operatorname{tr}((M'_{\tilde{u}} M'^*_{\tilde{n}} - M'_{\tilde{n}} M'^*_{\tilde{u}}) M^2_{ud} + (M'^*_{\tilde{n}} M'_{\tilde{u}} - M'^*_{\tilde{u}} M'_{\tilde{n}}) M^2_{en})}{2\mathrm{i}\ \operatorname{tr}(3(M^2_{ud})^2 + (M^2_{en})^2)} \;, \\
\gamma_V &= -\tfrac{1}{6}\operatorname{tr}(M'_{\tilde{n}} M'^*_{\tilde{n}}) \;, \qquad \epsilon_V = -\frac{\operatorname{tr}(M'_{\tilde{n}} M'^*_{\tilde{n}} \tilde{M}^2_V)}{\operatorname{tr}((\tilde{M}^2_V)^2)} \;, \\
\gamma_W &= -\tfrac{1}{12}\operatorname{tr}(M'_{\tilde{u}} M'^*_{\tilde{n}} + M'_{\tilde{n}} M'^*_{\tilde{u}}) \;, \qquad \epsilon_W = -\frac{\operatorname{tr}((M'_{\tilde{u}} M'^*_{\tilde{n}} + M'_{\tilde{n}} M'^*_{\tilde{u}})\tilde{M}^2_V)}{2\ \operatorname{tr}((\tilde{M}^2_V)^2)} \;, \\
\gamma'_W &= -\tfrac{1}{12\mathrm{i}}\operatorname{tr}(M'_{\tilde{u}} M'^*_{\tilde{n}} - M'_{\tilde{n}} M'^*_{\tilde{u}}) \;, \qquad \epsilon'_W = -\frac{\operatorname{tr}((M'_{\tilde{u}} M'^*_{\tilde{n}} - M'_{\tilde{n}} M'^*_{\tilde{u}})\tilde{M}^2_V)}{2\mathrm{i}\ \operatorname{tr}((\tilde{M}^2_V)^2)} \;, \\
\gamma_U &= -\tfrac{1}{6}\operatorname{tr}(M'_N M'^*_N) \;, \qquad \epsilon_U = -\frac{\operatorname{tr}(M'_N M'^*_N \tilde{M}^2_V)}{\operatorname{tr}((\tilde{M}^2_V)^2)} \;,
\end{aligned}
\tag{A.2}
$$

$$\tilde{\gamma}_U = -\tfrac{1}{12}\operatorname{tr}(M'_N M'^{\,*}_d + M'_d M'_N{}^*)\;, \qquad \tilde{\epsilon}_U = -\frac{\operatorname{tr}((M'_N M'^{\,*}_d + M'_d M'_N{}^*)\tilde{M}^2_V)}{2\ \operatorname{tr}((\tilde{M}^2_V)^2)}\;,$$
$$\tilde{\gamma}'_U = -\tfrac{1}{12\mathrm{i}}\operatorname{tr}(M'_N M'^{\,*}_d - M'_d M'_N{}^*)\;, \qquad \tilde{\epsilon}'_U = -\frac{\operatorname{tr}((M'_N M'^{\,*}_d - M'_d M'_N{}^*)\tilde{M}^2_V)}{2\mathrm{i}\ \operatorname{tr}((\tilde{M}^2_V)^2)}\;.$$

Appendix B. The Coefficients Occurring in the Higgs Potential

$$\begin{aligned}
&\mu_0 = \tfrac{1}{96}\operatorname{tr}(3M'_{10}{}^2 + M'^2_5)\;, && \mu_2 = \tfrac{1}{48}\operatorname{tr}(M'_{\tilde{n}} M'^{\,*}_{\tilde{n}})\;,\\
&\mu_1 = \operatorname{tr}(\tfrac{1}{16}M'_d M'^{\,*}_d + \tfrac{1}{12}M'_{\tilde{u}} M'^{\,*}_{\tilde{u}} + \tfrac{1}{48}M'_e M'^{\,*}_e)\;, && \mu_3 = \tfrac{1}{96}\operatorname{tr}(M'_N M'_N{}^*)\;,
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
\mu^{\mathrm{a}} &= \operatorname{tr}(10(\hat{M}^{10}_{aa})^2 + 5(\hat{M}^5_{aa})^2 + (\hat{M}^1_{aa})^2)\;,\\
\mu^{\mathrm{b}} &= \operatorname{tr}(10(\hat{M}^{10}_{bb})^2 + 5(\hat{M}^5_{bb})^2 + (\hat{M}^1_{bb})^2)\;,\\
\mu^{\mathrm{c}} &= \operatorname{tr}(10(\hat{M}^{10}_{nn})^2 + 5(\hat{M}^5_{nn})^2 + (\hat{M}^1_{nn})^2)\;,\\
\mu^{\mathrm{d}} &= \operatorname{tr}(20\hat{M}^{10}_{aa}\hat{M}^{10}_{bb} + 10\hat{M}^5_{aa}\hat{M}^5_{bb} + 2\hat{M}^1_{aa}\hat{M}^1_{bb})\;,\\
\mu^{\mathrm{e}} &= \operatorname{tr}(20\hat{M}^1_{aa}\hat{M}^{10}_{nn} + 10\hat{M}^5_{aa}\hat{M}^5_{nn} + 2\hat{M}^1_{aa}\hat{M}^1_{nn})\\
\mu^{\mathrm{f}} &= \operatorname{tr}(20\hat{M}^{10}_{bb}\hat{M}^{10}_{nn} + 10\hat{M}^5_{bb}\hat{M}^5_{nn} + 2\hat{M}^1_{bb}\hat{M}^1_{nn})\;,\\
\mu^{\mathrm{g}} &= \operatorname{tr}(\tfrac{3}{2}(M'_{10}M'_d + M'_d M_{10}{}^T)(M'_{10}M'_d + M'_d M_{10}{}^T)^* + 2(M'^{\,T}_5)^2 M'_e M'^{\,*}_e)\;,\\
\mu^{\mathrm{h}} &= \tfrac{1}{4}\operatorname{tr}((M'_{10}M'_d - M'_d M'^{\,T}_{10})(M'_{10}M'_d - M'_d M'^{\,T}_{10})^*)\;,
\end{aligned} \tag{B.2}$$

$$\begin{aligned}
&\mu^{\mathrm{i}} = \operatorname{tr}(2M'_{\tilde{u}} M'^{\,*}_{\tilde{u}} M'_{10}{}^2)\;, && \mu^{\mathrm{j}} = \operatorname{tr}(2M'^{\,*}_{\tilde{u}} M'_{\tilde{u}} M'^2_5)\;,\\
&\mu^{\mathrm{k}} = \operatorname{tr}(2M'_{\tilde{n}} M'^{\,*}_{\tilde{n}} M'_{10}{}^2)\;, && \mu^{\mathrm{l}} = \operatorname{tr}(2M'^{\,*}_{\tilde{n}} M'_{\tilde{n}} M'^2_5)\;,\\
&\mu^{\mathrm{m}} = \operatorname{tr}(4M'_{\tilde{u}} M'_5 M'^{\,*}_{\tilde{u}} M'_{10})\;, && \mu^{\mathrm{n}} = \operatorname{Re}(\operatorname{tr}(4M'_{\tilde{u}} M'^{\,*}_{\tilde{n}} M'_{10}{}^2))\;,\\
&\mu^{\mathrm{o}} = \operatorname{Im}(\operatorname{tr}(4M'_{\tilde{u}} M'^{\,*}_{\tilde{n}} M'_{10}{}^2))\;, && \mu^{\mathrm{p}} = \operatorname{Re}(\operatorname{tr}(4M'_{\tilde{u}} M'_5 M'^{\,*}_{\tilde{n}} M'_{10}))\;,\\
&\mu^{\mathrm{q}} = \operatorname{Im}(\operatorname{tr}(4M'_{\tilde{u}} M'_5 M'^{\,*}_{\tilde{n}} M'_{10}))\;, && \mu^{\mathrm{r}} = \operatorname{Re}(\operatorname{tr}(4M'^{\,*}_{\tilde{n}} M'_{\tilde{u}} M'^2_5))\;,\\
&\mu^{\mathrm{s}} = \operatorname{Im}(\operatorname{tr}(4M'^{\,*}_{\tilde{n}} M'_{\tilde{u}} M'^2_5))\;, && \mu^{\mathrm{t}} = \operatorname{tr}(4M'_{\tilde{n}} M'_5 M'^{\,*}_{\tilde{n}} M'_{10})\;,\\
&\mu^{\mathrm{u}} = \operatorname{tr}(2M'^{\,*}_{\tilde{n}} M'_{\tilde{n}} \bar{M}'_e M'^{\,T}_e)\;, && \mu^{\mathrm{v}} = \operatorname{tr}(2M'_{\tilde{n}} M'^{\,*}_{\tilde{n}} M'_d M'^{\,*}_d)\;,\\
&\mu^{\mathrm{w}} = \operatorname{tr}(2M'_{d\tilde{n}} M'_{d\tilde{n}}{}^*)\;,
\end{aligned}$$

$$\begin{aligned}
\breve{\mu}^{\mathrm{a}} &= \operatorname{tr}(10(\hat{M}^{10}_{cc})^2 + 5(\hat{M}^5_{cc})^2 + (\hat{M}^1_{cc})^2)\;,\\
\breve{\mu}^{\mathrm{b}} &= \operatorname{tr}(20\hat{M}^{10}_{cc}\hat{M}^{10}_{aa} + 10\hat{M}^5_{cc}\hat{M}^5_{aa} + 2\hat{M}^1_{cc}\hat{M}^1_{aa})\;,\\
\breve{\mu}^{\mathrm{c}} &= \operatorname{tr}(20\hat{M}^1_{cc}\hat{M}^{10}_{bb} + 10\hat{M}^5_{cc}\hat{M}^5_{bb} + 2\hat{M}^1_{cc}\hat{M}^1_{bb})\\
\breve{\mu}^{\mathrm{d}} &= \operatorname{tr}(20\hat{M}^{10}_{cc}\hat{M}^{10}_{nn} + 10\hat{M}^5_{cc}\hat{M}^5_{nn} + 2\hat{M}^1_{cc}\hat{M}^1_{nn})\;,\\
\breve{\mu}^{\mathrm{e}} &= \tfrac{1}{4}\operatorname{tr}((M'_{10}M'_N + M'_N M'^{\,T}_{10})(M'_{10}M'_N + M'_N M'^{\,T}_{10})^*)\;,\\
\breve{\mu}^{\mathrm{f}} &= \tfrac{1}{4}\operatorname{tr}((M'_{10}M'_N - M'_N M'^{\,T}_{10})(M'_{10}M'_N - M'_N M'^{\,T}_{10})^*)\;,\\
\breve{\mu}^{\mathrm{g}} &= \tfrac{1}{2}\operatorname{Re}(\operatorname{tr}((M'_{10}M'_N - M'_N M'^{\,T}_{10})(M'_{10}M'_d - M'_d M'^{\,T}_{10})^*))\;,\\
\breve{\mu}^{\mathrm{h}} &= \tfrac{1}{2}\operatorname{Im}(\operatorname{tr}((M'_{10}M'_N - M'_N M'^{\,T}_{10})(M'_{10}M'_d - M'_d M'^{\,T}_{10})^*))\;,
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
&\breve{\mu}^{\mathrm{i}} = \operatorname{tr}(2M'_{\tilde{n}} M'^{\,*}_{\tilde{n}} M'_N M'_N{}^*)\;, && \breve{\mu}^{\mathrm{j}} = \operatorname{Re}(\operatorname{tr}(4M'_{\tilde{n}} M'^{\,*}_{\tilde{n}} M'_N M'^{\,*}_d))\;,\\
&\breve{\mu}^{\mathrm{k}} = \operatorname{Im}(\operatorname{tr}(4M'_{\tilde{n}} M'^{\,*}_{\tilde{n}} M'_N M'^{\,*}_d))\;, && \breve{\mu}^{\mathrm{l}} = \operatorname{tr}(2M'_{Nu} M'_{Nu}{}^*)\;,\\
&\breve{\mu}^{\mathrm{m}} = \operatorname{Re}(\operatorname{tr}(4M'_{d\tilde{n}} M'_{Nu}{}^*))\;, && \breve{\mu}^{\mathrm{n}} = \operatorname{Im}(\operatorname{tr}(4M'_{d\tilde{n}} M'_{Nu}{}^*))\;,
\end{aligned}$$

$$\begin{aligned}
&\tilde{\mu}^{\mathrm{a}} = \operatorname{tr}(\tfrac{1}{3}(M^{10}_{aa})^2 + (M^5_{aa})^2)\;, && \tilde{\mu}^{\mathrm{b}} = \operatorname{tr}(3(\check{M}^{10}_{nn})^2 + (\check{M}^5_{nn})^2)\;,\\
&\tilde{\mu}^{\mathrm{c}} = \operatorname{tr}(3(M^{10}_{nn})^2 + (M^5_{nn})^2)\;, && \tilde{\mu}^{\mathrm{d}} = \operatorname{tr}(\tfrac{1}{3}(M^{10}_{\{un\}})^2 + (M^5_{\{un\}})^2)\;,
\end{aligned} \tag{B.4}$$

$$
\begin{aligned}
&\tilde\mu^{\mathrm{e}} = \mathrm{tr}\big(\tfrac{1}{3}(M^{10}_{[un]})^2 + (M^5_{[un]})^2\big)\;, &&\tilde\mu^{\mathrm{f}} = \mathrm{tr}\big(2M^{10}_{aa}\check M^{10}_{nn} + 2M^5_{aa}\check M^5_{nn}\big)\;,\\
&\tilde\mu^{\mathrm{g}} = \mathrm{tr}\big(2M^{10}_{aa}M^{10}_{nn} + 2M^5_{aa}M^5_{nn}\big)\;, &&\tilde\mu^{\mathrm{h}} = \mathrm{tr}\big(\tfrac{2}{3}M^{10}_{aa}M^{10}_{\{un\}} + 2M^5_{aa}M^5_{\{un\}}\big)\;,\\
&\tilde\mu^{\mathrm{i}} = \mathrm{tr}\big(\tfrac{2}{3}M^{10}_{aa}M^{10}_{[un]} + 2M^5_{aa}M^5_{[un]}\big)\;, &&\tilde\mu^{\mathrm{j}} = \mathrm{tr}\big(6\check M^{10}_{nn}M^{10}_{nn} + 2\check M^5_{nn}M^5_{nn}\big)\;,\\
&\tilde\mu^{\mathrm{k}} = \mathrm{tr}\big(2\check M^{10}_{nn}M^{10}_{\{un\}} + 2\check M^5_{nn}M^5_{\{un\}}\big)\;, &&\tilde\mu^{\mathrm{l}} = \mathrm{tr}\big(2\check M^{10}_{nn}M^{10}_{[un]} + 2\check M^5_{nn}M^5_{[un]}\big)\;,\\
&\tilde\mu^{\mathrm{m}} = \mathrm{tr}\big(2M^{10}_{nn}M^{10}_{\{un\}} + 2M^5_{nn}M^5_{\{un\}}\big)\;, &&\tilde\mu^{\mathrm{n}} = \mathrm{tr}\big(2M^{10}_{nn}M^{10}_{[un]} + 2M^5_{nn}M^5_{[un]}\big)\;,\\
&\tilde\mu^{\mathrm{o}} = \mathrm{tr}\big(\tfrac{2}{3}M^{10}_{\{un\}}M^{10}_{[un]} + 2M^5_{\{un\}}M^5_{[un]}\big)\;,\\
&\tilde\mu^{\mathrm{p}} = \mathrm{tr}\big((\tilde M^{10}_{nn})^2\big)\;, &&\tilde\mu^{\mathrm{q}} = \mathrm{tr}\big((\tilde M^{10}_{\{un\}})^2\big)\;,\\
&\tilde\mu^{\mathrm{r}} = \mathrm{tr}\big((\tilde M^{10}_{[un]})^2\big)\;, &&\tilde\mu^{\mathrm{s}} = \mathrm{tr}\big(2\tilde M^{10}_{nn}\tilde M^{10}_{\{un\}}\big)\;,\\
&\tilde\mu^{\mathrm{t}} = \mathrm{tr}\big(2\tilde M^{10}_{nn}\tilde M^{10}_{[un]}\big)\;, &&\tilde\mu^{\mathrm{u}} = \mathrm{tr}\big(2\tilde M^{10}_{\{un\}}\tilde M^{10}_{[un]}\big)\;,\\[1ex]
&\hat\mu^{\mathrm{a}} = \mathrm{tr}\big(3(M^{10}_{cc})^2 + (M^5_{cc})^2\big)\;, &&\hat\mu^{\mathrm{b}} = \mathrm{tr}\big(2M^{10}_{cc}M^{10}_{aa} + 2M^5_{cc}M^5_{aa}\big)\;,\\
&\hat\mu^{\mathrm{c}} = \mathrm{tr}\big(6M^{10}_{cc}\check M^{10}_{nn} + 2M^5_{cc}\check M^5_{nn}\big)\;, &&\hat\mu^{\mathrm{d}} = \mathrm{tr}\big(6M^{10}_{cc}M^{10}_{nn} + 2M^5_{cc}M^5_{nn}\big)\;,\\
&\hat\mu^{\mathrm{e}} = \mathrm{tr}\big(2M^{10}_{cc}M^{10}_{\{un\}} + 2M^5_{cc}M^5_{\{un\}}\big)\;, &&\hat\mu^{\mathrm{f}} = \mathrm{tr}\big(2M^{10}_{cc}M^{10}_{[un]} + 2M^5_{cc}M^5_{[un]}\big)\;,\\[1ex]
&\hat\mu^{\mathrm{g}} = \mathrm{tr}\big((\tilde M^{10}_{cc})^2\big)\;, &&\hat\mu^{\mathrm{h}} = \mathrm{tr}\big((\tilde M^{10}_{\{cd\}})^2\big)\;,\\
&\hat\mu^{\mathrm{i}} = \mathrm{tr}\big((\tilde M^{10}_{[cd]})^2\big)\;, &&\hat\mu^{\mathrm{j}} = \mathrm{tr}\big(2\tilde M^{10}_{cc}\tilde M^{10}_{\{cd\}}\big)\;,\\
&\hat\mu^{\mathrm{k}} = \mathrm{tr}\big(2\tilde M^{10}_{cc}\tilde M^{10}_{[cd]}\big)\;, &&\hat\mu^{\mathrm{l}} = \mathrm{tr}\big(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{[cd]}\big)\;,\\
&\hat\mu^{\mathrm{m}} = \mathrm{tr}\big(2\tilde M^{10}_{cc}\tilde M^{10}_{nn}\big)\;, &&\hat\mu^{\mathrm{n}} = \mathrm{tr}\big(2\tilde M^{10}_{cc}\tilde M^{10}_{\{un\}}\big)\;,\\
&\hat\mu^{\mathrm{o}} = \mathrm{tr}\big(2\tilde M^{10}_{cc}\tilde M^{10}_{[un]}\big)\;, &&\hat\mu^{\mathrm{p}} = \mathrm{tr}\big(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{nn}\big)\;,\\
&\hat\mu^{\mathrm{q}} = \mathrm{tr}\big(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{\{un\}}\big)\;, &&\hat\mu^{\mathrm{r}} = \mathrm{tr}\big(2\tilde M^{10}_{\{cd\}}\tilde M^{10}_{[un]}\big)\;,\\
&\hat\mu^{\mathrm{s}} = \mathrm{tr}\big(2\tilde M^{10}_{[cd]}\tilde M^{10}_{nn}\big)\;, &&\hat\mu^{\mathrm{t}} = \mathrm{tr}\big(2\tilde M^{10}_{[cd]}\tilde M^{10}_{\{un\}}\big)\;,\\
&\hat\mu^{\mathrm{u}} = \mathrm{tr}\big(2\tilde M^{10}_{[cd]}\tilde M^{10}_{[un]}\big)\;.
\end{aligned}
\tag{B.5}
$$

Appendix C. The Quadratic Terms of the Higgs Potential

$$
\begin{aligned}
\mathcal{L}_0 = \tfrac{1}{384}\Big\{&\\
&\tfrac{1}{12\mu_2+\mu_1}{\phi'_0}^2\big(8\mu^{\mathrm{b}}+1152\mu^{\mathrm{c}}+96\mu^{\mathrm{f}}+\tfrac{3072}{5}\tilde\mu^{\mathrm{b}}+\tfrac{1024}{15}\tilde\mu^{\mathrm{c}}+\tfrac{3072}{5}\tilde\mu^{\mathrm{d}}+\tfrac{1024}{5}\tilde\mu^{\mathrm{j}}-\tfrac{3072}{5}\tilde\mu^{\mathrm{k}}-\tfrac{1024}{5}\tilde\mu^{\mathrm{m}}\\
&\qquad+256\tilde\mu^{\mathrm{p}}+256\tilde\mu^{\mathrm{q}}-256\tilde\mu^{\mathrm{s}}\big)\\
&+\sqrt{\tfrac{6}{5}}\tfrac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_0\phi'_0\big(8\mu^{\mathrm{d}}+96\mu^{\mathrm{e}}-\tfrac{32}{3}\hat\mu^{\mathrm{c}}-\tfrac{32}{9}\hat\mu^{\mathrm{d}}+\tfrac{32}{3}\hat\mu^{\mathrm{e}}-\tfrac{80}{3}\hat\mu^{\mathrm{m}}+\tfrac{80}{3}\hat\mu^{\mathrm{n}}\\
&\qquad-512\tilde\mu^{\mathrm{b}}-\tfrac{512}{9}\tilde\mu^{\mathrm{c}}-512\tilde\mu^{\mathrm{d}}+\tfrac{32}{5}\tilde\mu^{\mathrm{f}}+\tfrac{32}{15}\tilde\mu^{\mathrm{g}}-\tfrac{32}{5}\tilde\mu^{\mathrm{h}}-\tfrac{512}{3}\tilde\mu^{\mathrm{j}}+512\tilde\mu^{\mathrm{k}}\\
&\qquad+\tfrac{512}{3}\tilde\mu^{\mathrm{m}}-\tfrac{1280}{3}\tilde\mu^{\mathrm{p}}-\tfrac{1280}{3}\tilde\mu^{\mathrm{q}}+\tfrac{1280}{3}\tilde\mu^{\mathrm{s}}\big)\\
&+\sqrt{3}\tfrac{1}{12\mu_2+\mu_1}\phi'_0\upsilon'_0\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}\big(384\mu^{\mathrm{c}}+16\mu^{\mathrm{f}}+\tfrac{448}{5}\tilde\mu^{\mathrm{b}}+\tfrac{1216}{45}\tilde\mu^{\mathrm{c}}+64\tilde\mu^{\mathrm{d}}+\tfrac{832}{15}\tilde\mu^{\mathrm{j}}-\tfrac{384}{5}\tilde\mu^{\mathrm{k}}-\tfrac{256}{5}\tilde\mu^{\mathrm{m}}\\
&\qquad+\tfrac{256}{3}\tilde\mu^{\mathrm{p}}+\tfrac{128}{3}\tilde\mu^{\mathrm{q}}-64\tilde\mu^{\mathrm{s}}\big)\\
&\qquad+\sqrt{\tfrac{\mu_2}{\mu_1}}\big(-32\mu^{\mathrm{b}}-192\mu^{\mathrm{f}}-\tfrac{6912}{5}\tilde\mu^{\mathrm{b}}+\tfrac{256}{5}\tilde\mu^{\mathrm{c}}-\tfrac{8448}{5}\tilde\mu^{\mathrm{d}}-\tfrac{768}{5}\tilde\mu^{\mathrm{j}}+1536\tilde\mu^{\mathrm{k}}\\
&\qquad+\tfrac{1024}{5}\tilde\mu^{\mathrm{m}}-512\tilde\mu^{\mathrm{q}}+256\tilde\mu^{\mathrm{s}}\big)\Big)\\
&+\sqrt{3}\tfrac{1}{\sqrt{\mu_2(12\mu_2+\mu_1)}}\phi'_0\upsilon_{45}\big(-32\tilde\mu^{\mathrm{l}}-\tfrac{32}{3}\tilde\mu^{\mathrm{n}}+32\tilde\mu^{\mathrm{o}}-\tfrac{64}{3}\tilde\mu^{\mathrm{t}}+\tfrac{64}{3}\tilde\mu^{\mathrm{u}}\big)\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_3(12\mu_2+\mu_1)}}\phi'_0\xi_0\big(4\check\mu^{\mathrm{c}}+48\check\mu^{\mathrm{d}}+\tfrac{64}{5}\hat\mu^{\mathrm{c}}+\tfrac{64}{15}\hat\mu^{\mathrm{d}}-\tfrac{64}{5}\hat\mu^{\mathrm{e}}+16\hat\mu^{\mathrm{m}}-16\hat\mu^{\mathrm{n}}\big)
\end{aligned}
\tag{C.1}
$$

$$
\begin{aligned}
&+\tfrac{1}{12\mu_2+\mu_1}\upsilon_0'^{\,2}\Big(\big(\tfrac{9}{4}\breve{\mu}^{l}-96\mu^{f}+24\mu^{h}+3\mu^{i}+3\mu^{j}+9\mu^{k}+9\mu^{l}-3\mu^{m}-9\mu^{t}+\tfrac{1728}{5}\tilde{\mu}^{b}+\tfrac{192}{5}\tilde{\mu}^{c}\\
&\qquad\qquad-192\tilde{\mu}^{d}-192\tilde{\mu}^{j}+\tfrac{384}{5}\tilde{\mu}^{k}+\tfrac{896}{5}\tilde{\mu}^{m}-128\tilde{\mu}^{q}+128\tilde{\mu}^{s}\big)\\
&\qquad\qquad+\tfrac{\mu_1}{\mu_2}\big(96\mu^{c}+\tfrac{3}{32}\breve{\mu}^{l}+\mu^{h}+\tfrac{1}{8}\mu^{i}+\tfrac{1}{8}\mu^{j}+\tfrac{3}{8}\mu^{k}+\tfrac{3}{8}\mu^{l}-\tfrac{1}{8}\mu^{m}-\tfrac{3}{8}\mu^{t}\\
&\qquad\qquad+\tfrac{184}{5}\tilde{\mu}^{b}+\tfrac{376}{45}\tilde{\mu}^{c}+8\tilde{\mu}^{d}+\tfrac{88}{15}\tilde{\mu}^{j}-16\tilde{\mu}^{k}-\tfrac{16}{3}\tilde{\mu}^{m}+\tfrac{64}{3}\tilde{\mu}^{p}+\tfrac{16}{3}\tilde{\mu}^{q}-\tfrac{32}{3}\tilde{\mu}^{s}\big)\\
&\qquad\qquad+\tfrac{\mu_2}{\mu_1}\big(96\mu^{b}+\tfrac{27}{2}\breve{\mu}^{l}+144\mu^{h}+18\mu^{i}+18\mu^{j}+54\mu^{k}+54\mu^{l}-18\mu^{m}-54\mu^{t}\\
&\qquad\qquad+\tfrac{31104}{5}\tilde{\mu}^{b}+\tfrac{384}{5}\tilde{\mu}^{c}+\tfrac{19584}{5}\tilde{\mu}^{d}-\tfrac{3456}{5}\tilde{\mu}^{j}-\tfrac{20736}{5}\tilde{\mu}^{k}+\tfrac{2304}{5}\tilde{\mu}^{m}+768\tilde{\mu}^{q}\big)\Big)\\
&+\sqrt{6}\tfrac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_3'\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}+12\sqrt{\tfrac{\mu_2}{\mu_1}}\Big)\big(2\mu^{h}+\tfrac{1}{4}\mu^{i}+\tfrac{1}{4}\mu^{j}-\tfrac{3}{4}\mu^{k}-\tfrac{3}{4}\mu^{l}-\tfrac{1}{4}\mu^{m}-\tfrac{1}{4}\mu^{n}\\
&\qquad\qquad\qquad+\tfrac{1}{2}\mu^{p}-\tfrac{1}{4}\mu^{r}+\tfrac{3}{4}\mu^{t}+48\tilde{\mu}^{b}-\tfrac{16}{9}\tilde{\mu}^{c}+16\tilde{\mu}^{d}-3\tilde{\mu}^{f}+\tfrac{1}{3}\tilde{\mu}^{g}+\tilde{\mu}^{h}\\
&\qquad\qquad\qquad+\tfrac{16}{3}\tilde{\mu}^{j}-32\tilde{\mu}^{k}+\hat{\mu}^{c}-\tfrac{1}{9}\hat{\mu}^{d}-\tfrac{1}{3}\hat{\mu}^{e}\big)\\
&+\tfrac{1}{\sqrt{\mu_2(12\mu_2+\mu_1)}}\upsilon_0'\upsilon_{45}\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}\big(-16\tilde{\mu}^{l}-\tfrac{16}{3}\tilde{\mu}^{n}+8\tilde{\mu}^{o}-\tfrac{32}{3}\tilde{\mu}^{t}+\tfrac{16}{3}\tilde{\mu}^{u}\big)+\sqrt{\tfrac{\mu_2}{\mu_1}}\big(-96\tilde{\mu}^{o}-64\tilde{\mu}^{u}\big)\Big)\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_3(12\mu_2+\mu_1)}}\xi_0\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}\big(8\breve{\mu}^{d}+\tfrac{14}{15}\hat{\mu}^{c}+\tfrac{38}{45}\hat{\mu}^{d}-\tfrac{2}{3}\hat{\mu}^{e}+\tfrac{8}{3}\hat{\mu}^{m}-\tfrac{4}{3}\hat{\mu}^{n}\big)\\
&\qquad\qquad\qquad+\sqrt{\tfrac{\mu_2}{\mu_1}}\big(-8\breve{\mu}^{c}-\tfrac{72}{5}\hat{\mu}^{c}+\tfrac{8}{5}\hat{\mu}^{d}+\tfrac{88}{5}\hat{\mu}^{e}+16\hat{\mu}^{n}\big)\Big)\\
&+\sqrt{\tfrac{2}{5}}\tfrac{1}{\sqrt{\mu_0(12\mu_2+\mu_1)}}\psi_0\upsilon_0'\Big(\sqrt{\tfrac{\mu_1}{\mu_2}}\big(48\mu^{e}-10\mu^{h}-\tfrac{5}{4}\mu^{i}-\tfrac{5}{4}\mu^{j}+\tfrac{15}{4}\mu^{k}+\tfrac{15}{4}\mu^{l}+\tfrac{5}{4}\mu^{m}+\tfrac{5}{4}\mu^{n}-\tfrac{5}{2}\mu^{p}\\
&\qquad\qquad\qquad+\tfrac{5}{4}\mu^{r}-\tfrac{15}{4}\mu^{t}-112\tilde{\mu}^{b}-\tfrac{304}{9}\tilde{\mu}^{c}-80\tilde{\mu}^{d}+\tfrac{7}{5}\tilde{\mu}^{f}+\tfrac{19}{15}\tilde{\mu}^{g}-\tilde{\mu}^{h}-\tfrac{208}{3}\tilde{\mu}^{j}\\
&\qquad\qquad\qquad+96\tilde{\mu}^{k}+64\tilde{\mu}^{m}-\tfrac{640}{3}\tilde{\mu}^{p}-\tfrac{320}{3}\tilde{\mu}^{q}+160\tilde{\mu}^{s}\\
&\qquad\qquad\qquad-\tfrac{7}{3}\hat{\mu}^{c}-\tfrac{19}{9}\hat{\mu}^{d}+\tfrac{5}{3}\hat{\mu}^{e}-\tfrac{40}{3}\hat{\mu}^{m}+\tfrac{20}{3}\hat{\mu}^{n}\big)\\
&\qquad\qquad\qquad+\sqrt{\tfrac{\mu_1}{\mu_2}}\big(-48\mu^{d}-120\mu^{h}-15\mu^{i}-15\mu^{j}+45\mu^{k}+45\mu^{l}+15\mu^{m}+15\mu^{n}\\
&\qquad\qquad\qquad-30\mu^{p}+15\mu^{r}-45\mu^{t}+1728\tilde{\mu}^{b}-64\tilde{\mu}^{c}+2112\tilde{\mu}^{d}-\tfrac{108}{5}\tilde{\mu}^{f}+\tfrac{12}{5}\tilde{\mu}^{g}\\
&\qquad\qquad\qquad+\tfrac{132}{5}\tilde{\mu}^{h}+192\tilde{\mu}^{j}-1920\tilde{\mu}^{k}-256\tilde{\mu}^{m}+1280\tilde{\mu}^{q}-640\tilde{\mu}^{s}\\
&\qquad\qquad\qquad+36\hat{\mu}^{c}-4\hat{\mu}^{d}-44\hat{\mu}^{e}-80\hat{\mu}^{n}\big)\Big)\\
&+\tfrac{1}{\mu_2}\upsilon_{45}^2\big(\tfrac{3}{32}\breve{\mu}^{l}+\mu^{h}+\tfrac{1}{8}\mu^{i}+\tfrac{1}{8}\mu^{j}+\tfrac{3}{8}\mu^{k}+\tfrac{3}{8}\mu^{l}-\tfrac{1}{8}\mu^{m}-\tfrac{3}{8}\mu^{t}+8\tilde{\mu}^{e}+\tfrac{16}{3}\tilde{\mu}^{r}\big)\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{\tfrac{2}{5}}\psi_0\upsilon_{45}\big(-\tfrac{7}{2}\mu^{o}+4\mu^{q}-\tfrac{1}{2}\mu^{s}-\tilde{\mu}^{i}+40\tilde{\mu}^{l}+\tfrac{40}{3}\tilde{\mu}^{n}-40\tilde{\mu}^{o}+\tfrac{160}{3}\tilde{\mu}^{t}-\tfrac{160}{3}\tilde{\mu}^{u}+\tfrac{5}{3}\hat{\mu}^{f}+\tfrac{20}{3}\hat{\mu}^{o}\big)\\
&+\sqrt{6}\tfrac{1}{\sqrt{\mu_0\mu_2}}\psi_3'\upsilon_{45}\big(-\tfrac{1}{2}\mu^{o}+\tfrac{1}{2}\mu^{s}+\tilde{\mu}^{i}-8\tilde{\mu}^{l}-\tfrac{8}{3}\tilde{\mu}^{n}+8\tilde{\mu}^{o}-\tfrac{1}{3}\hat{\mu}^{f}\big)\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\xi_0\upsilon_{45}\big(-\tfrac{2}{3}\hat{\mu}^{f}-\tfrac{4}{3}\hat{\mu}^{o}\big)\\
&+\tfrac{1}{\mu_0}\psi_0^2\big(\tfrac{48}{5}\mu^{a}+\tfrac{3}{5}\mu^{g}+10\mu^{h}+\tfrac{13}{5}\mu^{i}+\tfrac{7}{5}\mu^{j}+\tfrac{109}{5}\mu^{k}+\tfrac{31}{5}\mu^{l}-\tfrac{4}{5}\mu^{m}-7\mu^{n}+8\mu^{p}-\mu^{r}-\tfrac{52}{5}\mu^{t}+\tfrac{2}{25}\tilde{\mu}^{a}\\
&\qquad+128\tilde{\mu}^{b}+\tfrac{128}{9}\tilde{\mu}^{c}+128\tilde{\mu}^{d}-\tfrac{16}{5}\tilde{\mu}^{f}-\tfrac{16}{15}\tilde{\mu}^{g}+\tfrac{16}{5}\tilde{\mu}^{h}+\tfrac{128}{3}\tilde{\mu}^{j}-128\tilde{\mu}^{k}-\tfrac{128}{3}\tilde{\mu}^{m}+\tfrac{640}{3}\tilde{\mu}^{p}\\
&\qquad+\tfrac{640}{3}\tilde{\mu}^{q}-\tfrac{640}{3}\tilde{\mu}^{s}+\tfrac{48}{5}\breve{\mu}^{e}+\tfrac{2}{9}\hat{\mu}^{a}-\tfrac{2}{15}\hat{\mu}^{b}+\tfrac{16}{3}\hat{\mu}^{c}+\tfrac{16}{9}\hat{\mu}^{d}-\tfrac{16}{3}\hat{\mu}^{e}+\tfrac{10}{3}\hat{\mu}^{g}+\tfrac{80}{3}\hat{\mu}^{m}-\tfrac{80}{3}\hat{\mu}^{n}\big)\\
&+\tfrac{1}{\mu_0}\textstyle\sum_{i=1}^3\psi_i'^{\,2}\big(\mu^{g}+6\mu^{h}+3\mu^{i}+\mu^{j}+3\mu^{k}+9\mu^{l}+3\mu^{n}-3\mu^{r}+2\tilde{\mu}^{a}+128\tilde{\mu}^{b}+\tfrac{128}{9}\tilde{\mu}^{c}+128\tilde{\mu}^{d}-16\tilde{\mu}^{f}\\
&\qquad\qquad-\tfrac{16}{3}\tilde{\mu}^{g}+16\tilde{\mu}^{h}+\tfrac{128}{3}\tilde{\mu}^{j}-128\tilde{\mu}^{k}-\tfrac{128}{3}\tilde{\mu}^{m}+\tfrac{2}{9}\hat{\mu}^{a}-\tfrac{2}{3}\hat{\mu}^{b}+\tfrac{16}{3}\hat{\mu}^{c}+\tfrac{16}{9}\hat{\mu}^{d}-\tfrac{16}{3}\hat{\mu}^{e}\big)\\
&+\tfrac{1}{\mu_0}\textstyle\sum_{i=1}^8\psi_i^2\big(16\mu^{h}+2\mu^{i}+2\mu^{j}+2\mu^{k}+2\mu^{l}-2\mu^{m}+2\mu^{n}-4\mu^{p}+2\mu^{r}-2\mu^{t}+2\tilde{\mu}^{a}+128\tilde{\mu}^{b}\\
&\qquad\qquad+\tfrac{128}{9}\tilde{\mu}^{c}+128\tilde{\mu}^{d}+16\tilde{\mu}^{f}+\tfrac{16}{3}\tilde{\mu}^{g}-16\tilde{\mu}^{h}+\tfrac{128}{3}\tilde{\mu}^{j}-128\tilde{\mu}^{k}-\tfrac{128}{3}\tilde{\mu}^{m}+\tfrac{256}{3}\tilde{\mu}^{p}\\
&\qquad\qquad+\tfrac{256}{3}\tilde{\mu}^{q}-\tfrac{256}{3}\tilde{\mu}^{s}+\tfrac{2}{9}\hat{\mu}^{a}+\tfrac{2}{3}\hat{\mu}^{b}+\tfrac{16}{3}\hat{\mu}^{c}+\tfrac{16}{9}\hat{\mu}^{d}-\tfrac{16}{3}\hat{\mu}^{e}+\tfrac{4}{3}\hat{\mu}^{g}+\tfrac{32}{3}\hat{\mu}^{m}-\tfrac{32}{3}\hat{\mu}^{n}\big)\\
&+\sqrt{\tfrac{3}{5}}\psi_0\psi_3'\big(2\mu^{g}-20\mu^{h}+2\mu^{i}-2\mu^{j}+2\mu^{k}-18\mu^{l}+4\mu^{m}+2\mu^{n}-8\mu^{p}+6\mu^{r}+20\mu^{t}\big)\\
&+\sqrt{\tfrac{2}{\mu_0\mu_3}}\sqrt{\tfrac{3}{5}}\psi_0\xi_0\big(8\breve{\mu}^{b}-\tfrac{8}{9}\hat{\mu}^{a}+\tfrac{4}{15}\hat{\mu}^{b}-\tfrac{32}{3}\hat{\mu}^{c}-\tfrac{32}{9}\hat{\mu}^{d}+\tfrac{32}{3}\hat{\mu}^{e}-\tfrac{20}{3}\hat{\mu}^{g}-\tfrac{80}{3}\hat{\mu}^{m}+\tfrac{80}{3}\hat{\mu}^{n}\big)\\
&+\tfrac{2}{\mu_3}\xi_0^2\big(4\breve{\mu}^{a}+\tfrac{8}{15}\hat{\mu}^{a}+2\hat{\mu}^{g}\big)\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{6}\big(\psi_1'\upsilon_{18}+\psi_2'\upsilon_{63}\big)\big(2\mu^{h}+\tfrac{1}{4}\mu^{i}+\tfrac{1}{4}\mu^{j}-\tfrac{3}{4}\mu^{k}-\tfrac{3}{4}\mu^{l}-\tfrac{1}{4}\mu^{m}-\tfrac{1}{4}\mu^{n}+\tfrac{1}{2}\mu^{p}-\tfrac{1}{4}\mu^{r}+\tfrac{3}{4}\mu^{t}\\
&\qquad\qquad\qquad+48\tilde{\mu}^{b}-\tfrac{16}{9}\tilde{\mu}^{c}+16\tilde{\mu}^{d}-3\tilde{\mu}^{f}+\tfrac{1}{3}\tilde{\mu}^{g}+\tilde{\mu}^{h}+\tfrac{16}{3}\tilde{\mu}^{j}-32\tilde{\mu}^{k}\\
&\qquad\qquad\qquad+\hat{\mu}^{c}-\tfrac{1}{9}\hat{\mu}^{d}-\tfrac{1}{3}\hat{\mu}^{e}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\sqrt{6}(\psi'_1\upsilon_{63}-\psi'_2\upsilon_{18})(-\tfrac12\mu^{o}+\tfrac12\mu^{s}+\tilde\mu^{i}-8\tilde\mu^{l}-\tfrac83\tilde\mu^{n}+8\tilde\mu^{o}-\tfrac13\hat\mu^{f})\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\textstyle\sum_{i=1}^8\psi_i\upsilon_i\,(8\mu^{h}+\mu^{i}+\mu^{j}-3\mu^{k}-3\mu^{l}-\mu^{m}-\mu^{n}+2\mu^{p}-\mu^{r}+3\mu^{t}-64\tilde\mu^{b}-\tfrac{64}{9}\tilde\mu^{c}+64\tilde\mu^{d}\\
&\qquad\qquad\qquad-4\tilde\mu^{f}-\tfrac43\tilde\mu^{g}-4\tilde\mu^{h}-\tfrac{64}{3}\tilde\mu^{j}-\tfrac{64}{3}\tilde\mu^{p}+\tfrac{64}{3}\tilde\mu^{q}-\tfrac43\hat\mu^{c}-\tfrac49\hat\mu^{d}-\tfrac43\hat\mu^{e}-\tfrac43\hat\mu^{m}-\tfrac43\hat\mu^{n})\\
&+\tfrac{1}{\sqrt{\mu_0\mu_2}}\textstyle\sum_{i=1}^8\psi_i\upsilon_{i+45}(-2\mu^{o}+4\mu^{q}-2\mu^{s}-4\tilde\mu^{i}-32\tilde\mu^{l}-\tfrac{32}{3}\tilde\mu^{n}+32\tilde\mu^{o}-\tfrac{32}{3}\tilde\mu^{t}+\tfrac{32}{3}\tilde\mu^{u}-\tfrac43\hat\mu^{f}-\tfrac43\hat\mu^{o})\\
&+\tfrac{\sqrt2}{\sqrt{\mu_0\mu_3}}\textstyle\sum_{i=1}^8\psi_i\xi_{i+32}(-4\check\mu^{g}+2\hat\mu^{j}+16\hat\mu^{p}-16\hat\mu^{q})\\
&+\tfrac{\sqrt2}{\sqrt{\mu_0\mu_3}}\textstyle\sum_{i=1}^8\psi_i\xi_{i+81}(-4\check\mu^{h}+2\hat\mu^{k}+16\hat\mu^{s}-16\hat\mu^{t})\\
&+\tfrac{2}{\mu_1}\textstyle\sum_{i=1}^6\phi_i^2(\mu^{g}+8\mu^{h}+\tfrac{13}{4}\mu^{i}+\tfrac54\mu^{j}+\tfrac94\mu^{k}+\tfrac94\mu^{l}-\tfrac14\mu^{m}-\tfrac94\mu^{n}\\
&\qquad\qquad+\tfrac32\mu^{p}+\tfrac34\mu^{r}-\tfrac94\mu^{t}+\mu^{u}+\tfrac13\mu^{v}+\tfrac{32}{3}\mu^{w}+162\tilde\mu^{b}+2\tilde\mu^{c}\\
&\qquad\qquad+98\tilde\mu^{d}+2\tilde\mu^{e}-18\tilde\mu^{j}-126\tilde\mu^{k}+14\tilde\mu^{m}+\tfrac{64}{3}\tilde\mu^{q}+\tfrac{64}{3}\tilde\mu^{r}+2\hat\mu^{h}+2\hat\mu^{i})\\
&+\sqrt2\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{44}+\phi_2\xi_{45}+\phi_3\xi_{46}+\phi_4\xi_{93}+\phi_5\xi_{94}+\phi_6\xi_{95})\\
&\qquad\qquad(-3\hat\mu^{c}+\tfrac13\hat\mu^{d}+\tfrac73\hat\mu^{e}+2\hat\mu^{h}+2\hat\mu^{i}+\tfrac83\hat\mu^{n})\\
&+\sqrt2\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{93}+\phi_2\xi_{94}+\phi_3\xi_{95}-\phi_4\xi_{44}-\phi_5\xi_{45}-\phi_6\xi_{46})(\tfrac13\hat\mu^{f}+\tfrac83\hat\mu^{o})\\
&+\sqrt3\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{47}+\phi_2\xi_{48}+\phi_3\xi_{49}+\phi_4\xi_{96}+\phi_5\xi_{97}+\phi_6\xi_{98})\\
&\qquad\qquad(\tfrac83\check\mu^{g}+\tfrac49\check\mu^{j}-\tfrac49\check\mu^{m}+\tfrac23\hat\mu^{j}+\tfrac83\hat\mu^{q}-\tfrac83\hat\mu^{u})\\
&+\sqrt3\tfrac{2}{\sqrt{\mu_1\mu_3}}(\phi_1\xi_{96}+\phi_2\xi_{97}+\phi_3\xi_{98}-\phi_4\xi_{47}-\phi_5\xi_{48}-\phi_6\xi_{49})\\
&\qquad\qquad(\tfrac83\check\mu^{h}+\tfrac49\check\mu^{k}+\tfrac49\check\mu^{n}+\tfrac23\hat\mu^{k}+\tfrac83\hat\mu^{r}+\tfrac83\hat\mu^{t})\\
&+\sqrt2\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_9+\phi_2\upsilon_{10}+\phi_3\upsilon_{11}+\phi_4\upsilon_{54}+\phi_5\upsilon_{55}+\phi_6\upsilon_{56})\\
&\qquad\qquad(-4\mu^{h}-\tfrac12\mu^{i}-\tfrac12\mu^{j}-\tfrac32\mu^{k}-\tfrac32\mu^{l}+\tfrac12\mu^{m}+2\mu^{n}-2\mu^{p}+\tfrac32\mu^{t}+\mu^{u}+\tfrac13\mu^{v}-\tfrac{16}{3}\mu^{w}\\
&\qquad\qquad+18\tilde\mu^{b}-\tfrac{10}{3}\tilde\mu^{c}-14\tilde\mu^{d}+2\tilde\mu^{e}+14\tilde\mu^{j}+2\tilde\mu^{k}-\tfrac{38}{3}\tilde\mu^{m}+\tfrac{32}{3}\tilde\mu^{q}-\tfrac{32}{3}\tilde\mu^{r}-\tfrac{16}{3}\tilde\mu^{s})\\
&+\sqrt2\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{54}+\phi_2\upsilon_{55}+\phi_3\upsilon_{56}-\phi_4\upsilon_9-\phi_5\upsilon_{10}-\phi_6\upsilon_{11})\\
&\qquad\qquad\qquad(2\mu^{o}-2\mu^{q}+8\tilde\mu^{l}-\tfrac83\tilde\mu^{n}-8\tilde\mu^{o}-\tfrac{16}{3}\tilde\mu^{t}+\tfrac{32}{3}\tilde\mu^{u})\\
&+\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{12}+\phi_2\upsilon_{13}+\phi_3\upsilon_{14}+\phi_4\upsilon_{57}+\phi_5\upsilon_{58}+\phi_6\upsilon_{59})\\
&\qquad(2\mu^{u}+\tfrac23\mu^{v}+\tfrac{16}{3}\mu^{w}+36\tilde\mu^{b}+4\tilde\mu^{c}-28\tilde\mu^{d}+4\tilde\mu^{e}-20\tilde\mu^{j}+4\tilde\mu^{k}+12\tilde\mu^{m}-\tfrac{32}{3}\tilde\mu^{q}+\tfrac{32}{3}\tilde\mu^{r}+16\tilde\mu^{s})\\
&+\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{57}+\phi_2\upsilon_{58}+\phi_3\upsilon_{59}-\phi_4\upsilon_{12}-\phi_5\upsilon_{13}-\phi_6\upsilon_{14})(16\tilde\mu^{l}-16\tilde\mu^{o}+16\tilde\mu^{t}-\tfrac{32}{3}\tilde\mu^{u})\\
&+\sqrt2\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{39}+\phi_2\upsilon_{40}+\phi_3\upsilon_{41}+\phi_4\upsilon_{84}+\phi_5\upsilon_{85}+\phi_6\upsilon_{86})(-6\hat\mu^{p}+2\hat\mu^{q}+2\hat\mu^{u})\\
&+\sqrt2\tfrac{\sqrt2}{\sqrt{\mu_1\mu_2}}(\phi_1\upsilon_{84}+\phi_2\upsilon_{85}+\phi_3\upsilon_{86}-\phi_4\upsilon_{39}-\phi_5\upsilon_{40}-\phi_6\upsilon_{41})(-2\hat\mu^{r}-6\hat\mu^{s}+2\hat\mu^{t})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=1}^6\xi_i^2+\sum_{i=50}^{55}\xi_i^2)(4\check\mu^{e}+\check\mu^{l})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=7}^{12}\xi_i^2+\sum_{i=56}^{61}\xi_i^2)(4\check\mu^{e}+2\hat\mu^{h}+2\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=13}^{18}\xi_i^2+\sum_{i=62}^{67}\xi_i^2)(16\check\mu^{e}+2\hat\mu^{h}+2\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=19}^{24}\xi_i^2+\sum_{i=68}^{73}\xi_i^2)(4\check\mu^{e}+\tfrac12\check\mu^{l}+\hat\mu^{h}+\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=25}^{32}\xi_i^2+\sum_{i=74}^{81}\xi_i^2)(9\check\mu^{e}+\check\mu^{f}+\tfrac12\check\mu^{l}+\hat\mu^{h}+\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=33}^{40}\xi_i^2(9\check\mu^{e}+\check\mu^{f}+4\hat\mu^{h})+\sum_{i=82}^{89}\xi_i^2(9\check\mu^{e}+\check\mu^{f}+4\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=41}^{43}\xi_i^2+\sum_{i=90}^{92}\xi_i^2)(\check\mu^{e}+\check\mu^{f}+\tfrac83\check\mu^{i}+\tfrac13\check\mu^{l}+\tfrac19\hat\mu^{a}+\tfrac23\hat\mu^{g})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=44}^{46}\xi_i^2+\sum_{i=93}^{95}\xi_i^2)(\check\mu^{e}+\check\mu^{f}+\tfrac19\hat\mu^{a}+\tfrac23\hat\mu^{g}+\hat\mu^{h}+\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}(\textstyle\sum_{i=47}^{49}\xi_i^2+\sum_{i=96}^{98}\xi_i^2)(4\check\mu^{e}+\tfrac83\check\mu^{f}+\tfrac{16}{9}\check\mu^{i}+\tfrac{1}{18}\check\mu^{l}+\tfrac23\hat\mu^{g}+\hat\mu^{h}+\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^6(\xi_{i+6}\xi_{i+12}+\xi_{i+55}\xi_{i+61})(4\hat\mu^{h}-4\hat\mu^{i})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^6(\xi_{i+6}\xi_{i+61}-\xi_{i+55}\xi_{i+12})(-4\hat\mu^{l})\\
&+\tfrac{2}{\mu_3}\textstyle\sum_{i=1}^8\xi_{i+32}\xi_{i+81}(4\hat\mu^{l})\\
&+\sqrt6\tfrac{2}{\mu_3}(\xi_{44}\xi_{47}+\xi_{45}\xi_{48}+\xi_{46}\xi_{49}+\xi_{93}\xi_{96}+\xi_{94}\xi_{97}+\xi_{95}\xi_{98})(\tfrac23\hat\mu^{j})
\end{aligned}
$$

$$
\begin{aligned}
&+\sqrt{6}\tfrac{2}{\mu_3}(\xi_{44}\xi_{96}+\xi_{45}\xi_{97}+\xi_{46}\xi_{98}-\xi_{93}\xi_{47}-\xi_{94}\xi_{48}-\xi_{95}\xi_{49})(\tfrac{2}{3}\hat{\mu}^{\mathrm{k}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{6}(\xi_{i+18}\upsilon_{i+29}+\xi_{i+67}\upsilon_{i+74})(-\breve{\mu}^{\mathrm{m}}-2\hat{\mu}^{\mathrm{p}}-2\hat{\mu}^{\mathrm{q}}-2\hat{\mu}^{\mathrm{u}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{6}(\xi_{i+18}\upsilon_{i+74}-\xi_{i+67}\upsilon_{i+29})(-\breve{\mu}^{\mathrm{n}}-2\hat{\mu}^{\mathrm{r}}+2\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+24}\upsilon_{i+18}+\xi_{i+73}\upsilon_{i+63})(\breve{\mu}^{\mathrm{g}}-\breve{\mu}^{\mathrm{m}}-2\hat{\mu}^{\mathrm{p}}-2\hat{\mu}^{\mathrm{q}}-2\hat{\mu}^{\mathrm{u}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+24}\upsilon_{i+63}-\xi_{i+73}\upsilon_{i+18})(-\breve{\mu}^{\mathrm{h}}-\breve{\mu}^{\mathrm{n}}-2\hat{\mu}^{\mathrm{r}}+2\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+32}\upsilon_{i}+\xi_{i+81}\upsilon_{i+45})(-\breve{\mu}^{\mathrm{g}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}\textstyle\sum_{i=1}^{8}(\xi_{i+32}\upsilon_{i+45}-\xi_{i+81}\upsilon_{i})(\breve{\mu}^{\mathrm{h}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{15}+\xi_{42}\upsilon_{16}+\xi_{43}\upsilon_{17}+\xi_{90}\upsilon_{60}+\xi_{91}\upsilon_{61}+\xi_{92}\upsilon_{62})(\tfrac{2}{3}\hat{\mu}^{\mathrm{c}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{d}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{e}}-4\hat{\mu}^{\mathrm{m}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{60}+\xi_{42}\upsilon_{61}+\xi_{43}\upsilon_{62}-\xi_{90}\upsilon_{15}-\xi_{91}\upsilon_{16}-\xi_{92}\upsilon_{17})(\tfrac{2}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{36}+\xi_{42}\upsilon_{37}+\xi_{43}\upsilon_{38}+\xi_{90}\upsilon_{81}+\xi_{91}\upsilon_{82}+\xi_{92}\upsilon_{83})(-\breve{\mu}^{\mathrm{g}}+\tfrac{4}{3}\breve{\mu}^{\mathrm{j}}-\tfrac{1}{3}\breve{\mu}^{\mathrm{m}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{81}+\xi_{42}\upsilon_{82}+\xi_{43}\upsilon_{83}-\xi_{90}\upsilon_{36}-\xi_{91}\upsilon_{37}-\xi_{92}\upsilon_{38})(-\breve{\mu}^{\mathrm{h}}+\tfrac{4}{3}\breve{\mu}^{\mathrm{k}}+\tfrac{1}{3}\breve{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{42}+\xi_{42}\upsilon_{43}+\xi_{43}\upsilon_{44}+\xi_{90}\upsilon_{87}+\xi_{91}\upsilon_{88}+\xi_{92}\upsilon_{89})\\
&\qquad(-\tfrac{8}{3}\breve{\mu}^{\mathrm{i}}-\tfrac{1}{3}\breve{\mu}^{\mathrm{l}}+2\hat{\mu}^{\mathrm{c}}-\tfrac{2}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{m}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{41}\upsilon_{87}+\xi_{42}\upsilon_{88}+\xi_{43}\upsilon_{89}-\xi_{90}\upsilon_{42}-\xi_{91}\upsilon_{43}-\xi_{92}\upsilon_{44})(\tfrac{2}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{9}+\xi_{45}\upsilon_{10}+\xi_{46}\upsilon_{11}+\xi_{93}\upsilon_{54}+\xi_{94}\upsilon_{55}+\xi_{95}\upsilon_{56})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{c}}-\tfrac{5}{9}\hat{\mu}^{\mathrm{d}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{e}}-\tfrac{4}{3}\hat{\mu}^{\mathrm{m}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{n}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{54}+\xi_{45}\upsilon_{55}+\xi_{46}\upsilon_{56}-\xi_{93}\upsilon_{9}-\xi_{94}\upsilon_{10}-\xi_{95}\upsilon_{11})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{f}}+\tfrac{4}{3}\hat{\mu}^{\mathrm{o}})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{12}+\xi_{45}\upsilon_{13}+\xi_{46}\upsilon_{14}+\xi_{93}\upsilon_{57}+\xi_{94}\upsilon_{58}+\xi_{95}\upsilon_{59})\\
&\qquad(-\tfrac{1}{3}\hat{\mu}^{\mathrm{c}}+\tfrac{1}{3}\hat{\mu}^{\mathrm{d}}-\tfrac{1}{3}\hat{\mu}^{\mathrm{e}}+2\hat{\mu}^{\mathrm{m}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{n}})\\
&+\sqrt{2}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{57}+\xi_{45}\upsilon_{58}+\xi_{46}\upsilon_{59}-\xi_{93}\upsilon_{12}-\xi_{94}\upsilon_{13}-\xi_{95}\upsilon_{14})(-\tfrac{1}{3}\hat{\mu}^{\mathrm{f}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{39}+\xi_{45}\upsilon_{40}+\xi_{46}\upsilon_{41}+\xi_{93}\upsilon_{84}+\xi_{94}\upsilon_{85}+\xi_{95}\upsilon_{86})(-\breve{\mu}^{\mathrm{g}}-6\hat{\mu}^{\mathrm{p}}+2\hat{\mu}^{\mathrm{q}}+2\hat{\mu}^{\mathrm{u}})\\
&+\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{44}\upsilon_{84}+\xi_{45}\upsilon_{85}+\xi_{46}\upsilon_{86}-\xi_{93}\upsilon_{39}-\xi_{94}\upsilon_{40}-\xi_{95}\upsilon_{41})(-\breve{\mu}^{\mathrm{h}}-2\hat{\mu}^{\mathrm{r}}-6\hat{\mu}^{\mathrm{s}}+2\hat{\mu}^{\mathrm{t}})\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{9}+\xi_{48}\upsilon_{10}+\xi_{49}\upsilon_{11}+\xi_{96}\upsilon_{54}+\xi_{97}\upsilon_{55}+\xi_{98}\upsilon_{56})\\
&\qquad(-\tfrac{2}{3}\breve{\mu}^{\mathrm{g}}+\tfrac{2}{9}\breve{\mu}^{\mathrm{j}}+\tfrac{1}{9}\breve{\mu}^{\mathrm{m}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{p}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{q}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{u}})\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{54}+\xi_{48}\upsilon_{55}+\xi_{49}\upsilon_{56}-\xi_{96}\upsilon_{9}-\xi_{97}\upsilon_{10}-\xi_{98}\upsilon_{11})\\
&\qquad(\tfrac{2}{3}\breve{\mu}^{\mathrm{h}}-\tfrac{2}{9}\breve{\mu}^{\mathrm{k}}+\tfrac{1}{9}\breve{\mu}^{\mathrm{n}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{r}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{s}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{t}})\\
&+\sqrt{3}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{12}+\xi_{48}\upsilon_{13}+\xi_{49}\upsilon_{14}+\xi_{96}\upsilon_{57}+\xi_{97}\upsilon_{58}+\xi_{98}\upsilon_{59})(\tfrac{4}{9}\breve{\mu}^{\mathrm{j}}-\tfrac{1}{9}\breve{\mu}^{\mathrm{m}}+2\hat{\mu}^{\mathrm{p}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{q}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{u}})\\
&+\sqrt{3}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{57}+\xi_{48}\upsilon_{58}+\xi_{49}\upsilon_{59}-\xi_{96}\upsilon_{12}-\xi_{97}\upsilon_{13}-\xi_{98}\upsilon_{14})\\
&\qquad(-\tfrac{4}{9}\breve{\mu}^{\mathrm{k}}-\tfrac{1}{9}\breve{\mu}^{\mathrm{n}}-\tfrac{2}{3}\hat{\mu}^{\mathrm{r}}-2\hat{\mu}^{\mathrm{s}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{t}})\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{39}+\xi_{48}\upsilon_{40}+\xi_{49}\upsilon_{41}+\xi_{96}\upsilon_{84}+\xi_{97}\upsilon_{85}+\xi_{98}\upsilon_{86})(-2\hat{\mu}^{\mathrm{m}}+\tfrac{2}{3}\hat{\mu}^{\mathrm{n}})\\
&+\sqrt{6}\tfrac{\sqrt{2}}{\sqrt{\mu_2\mu_3}}(\xi_{47}\upsilon_{84}+\xi_{48}\upsilon_{85}+\xi_{49}\upsilon_{86}-\xi_{96}\upsilon_{39}-\xi_{97}\upsilon_{40}-\xi_{98}\upsilon_{41})(-\tfrac{2}{3}\hat{\mu}^{\mathrm{o}})\\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=1}^{8}\upsilon_i^2(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}}-\tfrac{9}{8}\mu^{\mathrm{t}}\\
&\qquad+8\tilde{\mu}^{\mathrm{b}}+\tfrac{8}{9}\tilde{\mu}^{\mathrm{c}}+8\tilde{\mu}^{\mathrm{d}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{j}}+8\tilde{\mu}^{\mathrm{k}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{m}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{s}})\\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=46}^{53}\upsilon_i^2(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{9}{8}\mu^{\mathrm{k}}+\tfrac{9}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{3}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}-\tfrac{3}{8}\mu^{\mathrm{r}}-\tfrac{9}{8}\mu^{\mathrm{t}}+8\tilde{\mu}^{\mathrm{e}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{r}})\\
&+\tfrac{1}{\mu_2}\textstyle\sum_{i=1}^{8}\upsilon_i\upsilon_{i+45}(8\tilde{\mu}^{\mathrm{l}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{n}}+8\tilde{\mu}^{\mathrm{o}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{t}}+\tfrac{16}{3}\tilde{\mu}^{\mathrm{u}})\\
&+\tfrac{1}{\mu_2}(\textstyle\sum_{i=9}^{11}\upsilon_i^2+\sum_{i=54}^{56}\upsilon_i^2)(\mu^{\mathrm{h}}+\tfrac{1}{8}\mu^{\mathrm{i}}+\tfrac{1}{8}\mu^{\mathrm{j}}+\tfrac{29}{8}\mu^{\mathrm{k}}+\tfrac{5}{8}\mu^{\mathrm{l}}-\tfrac{1}{8}\mu^{\mathrm{m}}-\tfrac{5}{8}\mu^{\mathrm{n}}+\tfrac{3}{4}\mu^{\mathrm{p}}-\tfrac{1}{8}\mu^{\mathrm{r}}-\tfrac{9}{8}\mu^{\mathrm{t}}\\
&\qquad+\tfrac{1}{2}\mu^{\mathrm{u}}+\tfrac{1}{6}\mu^{\mathrm{v}}+\tfrac{4}{3}\mu^{\mathrm{w}}+\tilde{\mu}^{\mathrm{b}}+\tfrac{25}{9}\tilde{\mu}^{\mathrm{c}}+\tilde{\mu}^{\mathrm{d}}+\tilde{\mu}^{\mathrm{e}}+\tfrac{5}{3}\tilde{\mu}^{\mathrm{j}}+\tilde{\mu}^{\mathrm{k}}+\tfrac{5}{3}\tilde{\mu}^{\mathrm{m}}\\
&\qquad+\tfrac{8}{3}\tilde{\mu}^{\mathrm{p}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{q}}+\tfrac{8}{3}\tilde{\mu}^{\mathrm{r}}-\tfrac{8}{3}\tilde{\mu}^{\mathrm{s}})
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=12}^{14}\upsilon_i^2+\sum_{i=57}^{59}\upsilon_i^2\big)\big(\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{9}{8}\mu^{\rm k}+\tfrac{1}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm n}+\tfrac{1}{4}\mu^{\rm p}+\tfrac{1}{8}\mu^{\rm r}+\tfrac{3}{8}\mu^{\rm t}+\mu^{\rm u}\\
&\qquad+\tfrac{1}{3}\mu^{\rm v}+\tfrac{2}{3}\mu^{\rm w}+2\tilde\mu^{\rm b}+2\tilde\mu^{\rm c}+2\tilde\mu^{\rm d}+2\tilde\mu^{\rm e}-2\tilde\mu^{\rm j}+2\tilde\mu^{\rm k}-2\tilde\mu^{\rm m}\\
&\qquad+12\tilde\mu^{\rm p}+\tfrac{4}{3}\tilde\mu^{\rm q}+\tfrac{4}{3}\tilde\mu^{\rm r}-4\tilde\mu^{\rm s}\big)\\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=15}^{17}\upsilon_i^2+\sum_{i=60}^{62}\upsilon_i^2\big)\big(\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{9}{8}\mu^{\rm k}+\tfrac{1}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm n}+\tfrac{1}{4}\mu^{\rm p}+\tfrac{1}{8}\mu^{\rm r}+\tfrac{3}{8}\mu^{\rm t}+4\tilde\mu^{\rm b}\\
&\qquad+4\tilde\mu^{\rm c}+4\tilde\mu^{\rm d}+4\tilde\mu^{\rm e}-4\tilde\mu^{\rm j}+4\tilde\mu^{\rm k}-4\tilde\mu^{\rm m}+24\tilde\mu^{\rm p}+\tfrac{8}{3}\tilde\mu^{\rm q}+\tfrac{8}{3}\tilde\mu^{\rm r}-8\tilde\mu^{\rm s}\big)\\
&+\tfrac{1}{\mu_2}\big(\upsilon_{18}^2+\upsilon_{63}^2\big)\big(\tfrac{3}{32}\check\mu^{\rm l}+\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{3}{8}\mu^{\rm k}+\tfrac{3}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm t}+\tfrac{3}{2}\mu^{\rm u}+\tfrac{1}{2}\mu^{\rm v}\\
&\qquad+27\tilde\mu^{\rm b}+\tfrac{1}{3}\tilde\mu^{\rm c}+3\tilde\mu^{\rm d}+3\tilde\mu^{\rm e}-3\tilde\mu^{\rm j}-9\tilde\mu^{\rm k}+\tilde\mu^{\rm m}\big)\\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=19}^{26}\upsilon_i^2+\sum_{i=64}^{71}\upsilon_i^2\big)\big(\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{9}{8}\mu^{\rm k}+\tfrac{9}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm n}+\tfrac{3}{4}\mu^{\rm p}-\tfrac{3}{8}\mu^{\rm r}-\tfrac{9}{8}\mu^{\rm t}+2\mu^{\rm w}\\
&\qquad+4\tilde\mu^{\rm p}+4\tilde\mu^{\rm q}+4\tilde\mu^{\rm r}+4\tilde\mu^{\rm s}\big)\\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=27}^{29}\upsilon_i^2+\sum_{i=72}^{74}\upsilon_i^2\big)\big(\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{9}{8}\mu^{\rm k}+\tfrac{1}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{3}{8}\mu^{\rm n}+\tfrac{1}{4}\mu^{\rm p}\\
&\qquad+\tfrac{1}{8}\mu^{\rm r}+\tfrac{3}{8}\mu^{\rm t}+2\mu^{\rm u}+\tfrac{2}{3}\mu^{\rm v}+\tfrac{4}{3}\mu^{\rm w}\big)\\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=30}^{35}\upsilon_i^2+\sum_{i=75}^{80}\upsilon_i^2\big)\big(\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{49}{8}\mu^{\rm k}+\tfrac{9}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{7}{8}\mu^{\rm n}+\tfrac{5}{4}\mu^{\rm p}-\tfrac{3}{8}\mu^{\rm r}-\tfrac{21}{8}\mu^{\rm t}\\
&\qquad+2\mu^{\rm w}+4\tilde\mu^{\rm p}+4\tilde\mu^{\rm q}+4\tilde\mu^{\rm r}+4\tilde\mu^{\rm s}\big)\\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=36}^{38}\upsilon_i^2+\sum_{i=82}^{84}\upsilon_i^2\big)\big(\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{49}{8}\mu^{\rm k}+\tfrac{1}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{7}{8}\mu^{\rm n}\\
&\qquad+\tfrac{3}{4}\mu^{\rm p}+\tfrac{1}{8}\mu^{\rm r}+\tfrac{7}{8}\mu^{\rm t}+2\mu^{\rm u}+\tfrac{2}{3}\mu^{\rm v}+\tfrac{4}{3}\mu^{\rm w}\big)\\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=39}^{41}\upsilon_i^2+\sum_{i=84}^{86}\upsilon_i^2\big)\big(\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{49}{8}\mu^{\rm k}+\tfrac{1}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}-\tfrac{7}{8}\mu^{\rm n}+\tfrac{3}{4}\mu^{\rm p}+\tfrac{1}{8}\mu^{\rm r}+\tfrac{7}{8}\mu^{\rm t}\\
&\qquad+2\mu^{\rm w}+36\tilde\mu^{\rm p}+4\tilde\mu^{\rm q}+4\tilde\mu^{\rm r}-12\tilde\mu^{\rm s}\big)\\
&+\tfrac{1}{\mu_2}\big(\textstyle\sum_{i=42}^{44}\upsilon_i^2+\sum_{i=87}^{89}\upsilon_i^2\big)\big(\tfrac{2}{3}\check\mu^{\rm i}+\tfrac{1}{12}\check\mu^{\rm l}+\mu^{\rm h}+\tfrac{1}{8}\mu^{\rm i}+\tfrac{1}{8}\mu^{\rm j}+\tfrac{1}{8}\mu^{\rm k}+\tfrac{9}{8}\mu^{\rm l}-\tfrac{1}{8}\mu^{\rm m}+\tfrac{1}{8}\mu^{\rm n}\\
&\qquad+\tfrac{1}{4}\mu^{\rm p}-\tfrac{3}{8}\mu^{\rm r}+\tfrac{3}{8}\mu^{\rm t}+36\tilde\mu^{\rm b}+\tfrac{4}{9}\tilde\mu^{\rm c}+4\tilde\mu^{\rm d}+4\tilde\mu^{\rm e}-4\tilde\mu^{\rm j}\\
&\qquad-12\tilde\mu^{\rm k}+\tfrac{4}{3}\tilde\mu^{\rm m}+\tfrac{8}{3}\tilde\mu^{\rm p}+\tfrac{8}{3}\tilde\mu^{\rm q}+\tfrac{8}{3}\tilde\mu^{\rm r}+\tfrac{8}{3}\tilde\mu^{\rm s}\big)\\
&+\sqrt{2}\tfrac{1}{\mu_2}\big(\upsilon_9\upsilon_{12}+\upsilon_{10}\upsilon_{13}+\upsilon_{11}\upsilon_{14}+\upsilon_{54}\upsilon_{57}+\upsilon_{55}\upsilon_{58}+\upsilon_{56}\upsilon_{59}\big)\\
&\qquad\big(\mu^{\rm u}+\tfrac{1}{3}\mu^{\rm v}-\tfrac{4}{3}\mu^{\rm w}+2\tilde\mu^{\rm b}-\tfrac{10}{3}\tilde\mu^{\rm c}+2\tilde\mu^{\rm d}+2\tilde\mu^{\rm e}+\tfrac{2}{3}\tilde\mu^{\rm j}+2\tilde\mu^{\rm k}+\tfrac{2}{3}\tilde\mu^{\rm m}-8\tilde\mu^{\rm p}-\tfrac{8}{3}\tilde\mu^{\rm q}-\tfrac{8}{3}\tilde\mu^{\rm r}+\tfrac{16}{3}\tilde\mu^{\rm s}\big)\\
&+\sqrt{2}\tfrac{1}{\mu_2}\big(\upsilon_9\upsilon_{57}+\upsilon_{10}\upsilon_{58}+\upsilon_{11}\upsilon_{59}-\upsilon_{12}\upsilon_{54}-\upsilon_{13}\upsilon_{55}-\upsilon_{14}\upsilon_{56}\big)\big(\tfrac{8}{3}\tilde\mu^{\rm n}-\tfrac{8}{3}\tilde\mu^{\rm t}\big)\\
&+\tfrac{1}{\mu_2}\big(\upsilon_{15}\upsilon_{42}+\upsilon_{16}\upsilon_{43}+\upsilon_{17}\upsilon_{54}+\upsilon_{60}\upsilon_{87}+\upsilon_{61}\upsilon_{88}+\upsilon_{62}\upsilon_{89}\big)\\
&\qquad\big(24\tilde\mu^{\rm b}+\tfrac{8}{3}\tilde\mu^{\rm c}-8\tilde\mu^{\rm d}+8\tilde\mu^{\rm e}-\tfrac{40}{3}\tilde\mu^{\rm j}+8\tilde\mu^{\rm k}+\tfrac{8}{3}\tilde\mu^{\rm m}+16\tilde\mu^{\rm p}-\tfrac{16}{3}\tilde\mu^{\rm q}+\tfrac{16}{3}\tilde\mu^{\rm r}+\tfrac{16}{3}\tilde\mu^{\rm s}\big)\\
&+\tfrac{1}{\mu_2}\big(\upsilon_{15}\upsilon_{87}+\upsilon_{16}\upsilon_{88}+\upsilon_{17}\upsilon_{89}-\upsilon_{60}\upsilon_{42}-\upsilon_{61}\upsilon_{43}-\upsilon_{62}\upsilon_{44}\big)\big(-8\tilde\mu^{\rm l}-\tfrac{8}{3}\tilde\mu^{\rm n}+8\tilde\mu^{\rm o}-\tfrac{16}{3}\tilde\mu^{\rm t}+\tfrac{16}{3}\tilde\mu^{\rm u}\big)\\
&+\tfrac{1}{\mu_2}\big(\upsilon_{36}\upsilon_{42}+\upsilon_{37}\upsilon_{43}+\upsilon_{38}\upsilon_{54}+\upsilon_{81}\upsilon_{87}+\upsilon_{82}\upsilon_{88}+\upsilon_{83}\upsilon_{89}\big)\big(-\tfrac{2}{3}\check\mu^{\rm j}+\tfrac{1}{6}\check\mu^{\rm m}\big)\\
&+\tfrac{1}{\mu_2}\big(\upsilon_{36}\upsilon_{87}+\upsilon_{37}\upsilon_{88}+\upsilon_{38}\upsilon_{89}-\upsilon_{81}\upsilon_{42}-\upsilon_{82}\upsilon_{43}-\upsilon_{83}\upsilon_{44}\big)\big(\tfrac{2}{3}\check\mu^{\rm k}+\tfrac{1}{6}\check\mu^{\rm n}\big)\big\}+I.T\ .
\end{aligned}
$$

References

[1] A. H. Chamseddine, G. Felder and J. Fröhlich, *Grand Unification in Non-Commutative Geometry*, Nucl. Phys. B **395** (1992) 672–698.

[2] A. H. Chamseddine, G. Felder and J. Fröhlich, *Unified Gauge Theories in Non-Commutative Geometry*, Phys. Lett. B **296** (1992) 109–116.

[3] A. H. Chamseddine and J. Fröhlich, *SO*(10) *Unification in Non–Commutative Geometry*, Phys. Rev. D **50** (1994) 2893–2907.

[4] A. Connes, *Non commutative geometry*, Academic Press, New York 1994.

[5] A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. **36** (1995) 6194–6231.

[6] A. Connes and J. Lott, *The Metric Aspect of Noncommutative Geometry*, Proceedings of 1991 Cargèse Summer Conference, ed. by J. Fröhlich et al, Plenum, New York 1992.

[7] J.-P. Derendinger, J. E. Kim and D. V. Nanopoulos, *Anti–*SU(5), Phys. Lett. B **139** (1984) 170–175.

[8] B. Iochum, D. Kastler and T. Schücker, *Fuzzy Mass Relations in the Standard Model*, preprint hep-th/9507150.

[9] P. Langacker, *Grand Unified Theories and Proton Decay*, Phys. Rep. **72** (1981) 185–385.

[10] F. Lizzi, G. Mangano, G. Miele and G. Sparano, *Constraints on Unified Gauge Theories from Noncommutative Geometry*, preprint hep-th/9603095.

[11] C. P. Martín, J. M. Gracia–Bondía and J. C. Varilly, *The standard model as a noncommutative geometry: the low energy regime*, preprint hep-th/9605001.

[12] R. Wulkenhaar, *Non–Associative Geometry — Unified Models Based on L–Cycles*, Ph.D. thesis, Leipzig 1997.

[13] R. Wulkenhaar, *Non–Commutative Geometry with Graded Differential Lie Algebras*, to appear in J. Math. Phys. (preprint `hep-th/9607094` under the title *'The Mathematical Footing of Non–associative Geometry'*)

[14] R. Wulkenhaar, *The Standard Model within Non–Associative Geometry*, Phys. Lett. B **390** (1997) 119–127.